

## Laplace transforms and second-order linear equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives:  $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$ . What does this rule say about  $\mathcal{L}\left[\frac{d^2y}{dt^2}\right]$ ?

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let's determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler's formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. *Mathematica* tells us that

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Let's use this fact to determine  $\mathcal{L}[\sin \omega t]$  and  $\mathcal{L}[\cos \omega t]$ .

Now that we know the transforms of sine and cosine, let's see how we use them.

**Example.** Compute

$$\mathcal{L}^{-1} \left[ \frac{2s + 1}{s^2 + 9} \right].$$

Now for a little practice with the third rule for transforms:

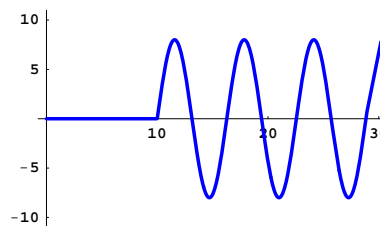
**Example.** Compute

$$\mathcal{L}^{-1} \left[ \frac{8e^{-10s}}{(s^2 + 9)(s^2 + 1)} \right].$$

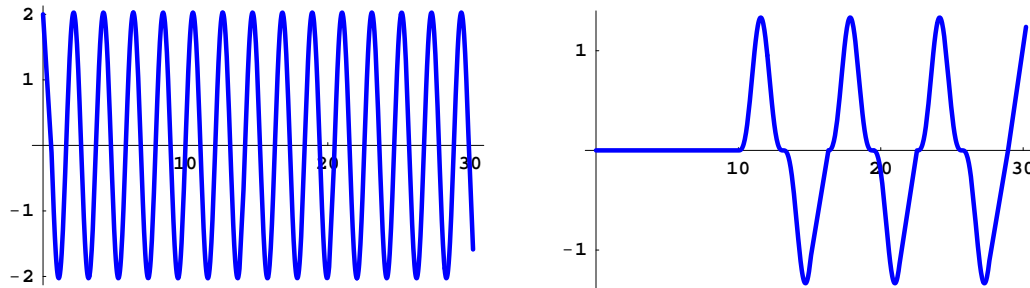
Now let's use what we have learned to solve the initial-value problem

$$\frac{d^2y}{dt^2} + 9y = 8u_{10}(t) \sin(t - 10), \quad y(0) = 2, \quad y'(0) = 1.$$

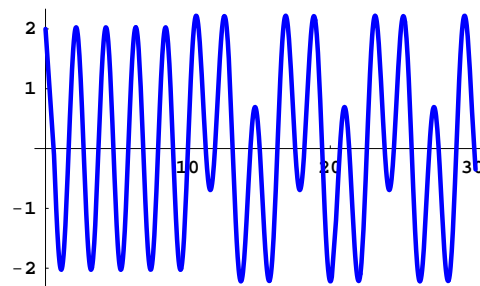
Here is the graph of the forcing function  $8u_{10}(t) \sin(t - 10)$ :



Here are the graphs of the two functions that combine to give us the desired solution.



Here is the graph of the solution



The second-order equation that we just considered is undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

**Shifting the  $s$ -axis.** Let  $Y(s)$  denote the Laplace transform  $\mathcal{L}[y(t)]$ . Then

$$\mathcal{L}[e^{at} y(t)] =$$

**Example 1.** Calculate  $\mathcal{L}[e^{-2t} \cos 3t]$ .

**Example 2.** Calculate  $\mathcal{L}^{-1} \left[ \frac{2s + 7}{s^2 + 4s + 7} \right]$ .