

More on undamped sinusoidal forcing

Last class we computed the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

assuming  $\omega^2 \neq 3$ , and we got  $y(t) = k_1 \cos \sqrt{3}t + k_2 \sin \sqrt{3}t + \frac{1}{3 - \omega^2} \cos \omega t$ .

Then, in order to understand the behavior of the solutions, we restricted our attention to those solutions that satisfy the initial condition  $(y(0), y'(0)) = (0, 0)$ . Solving for  $k_1$  and  $k_2$  yields

$$y(t) = \frac{1}{3 - \omega^2} (\cos \omega t - \cos \sqrt{3}t).$$

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency  $\omega$ . The following trig identity helps us interpret what we see in the animation.

Trig identity:

$$\cos at - \cos bt = -2 (\sin \alpha t) (\sin \beta t)$$

where

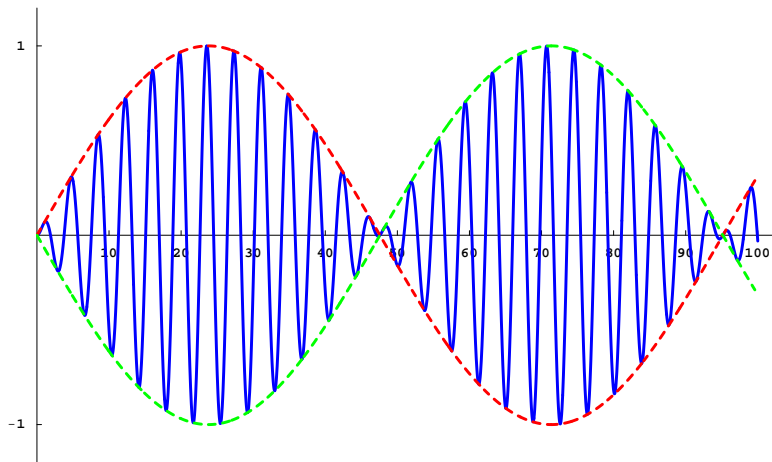
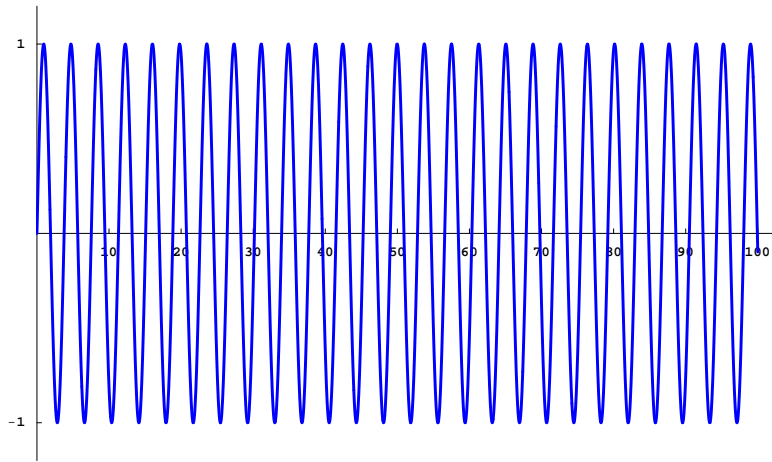
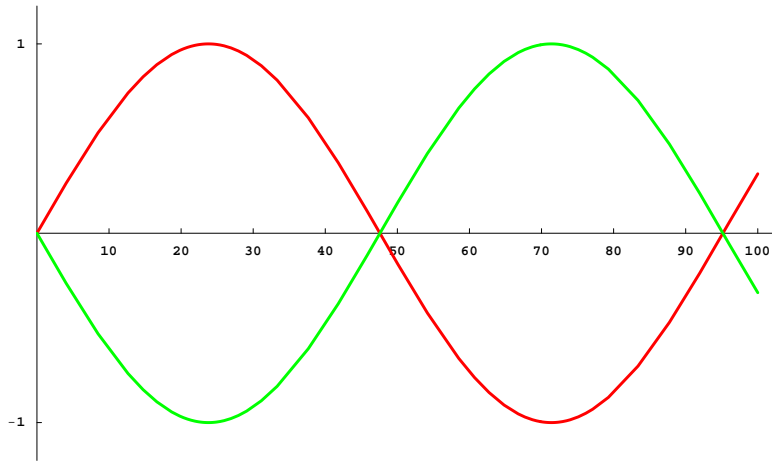
$$\alpha = \frac{a + b}{2} \quad \text{and} \quad \beta = \frac{a - b}{2}.$$

The number  $\alpha$  is the average of  $a$  and  $b$ , and  $\beta$  is called the *half-difference* of  $a$  and  $b$ .

**Example.** Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}t$$

where  $\omega = 1.6$ .



Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition  $(y(0), y'(0)) = (0, 0)$ . If  $\omega \neq \pm\sqrt{3}$ , the solution is

$$y(t) = \frac{1}{3 - \omega^2} (\cos \omega t - \cos \sqrt{3} t).$$

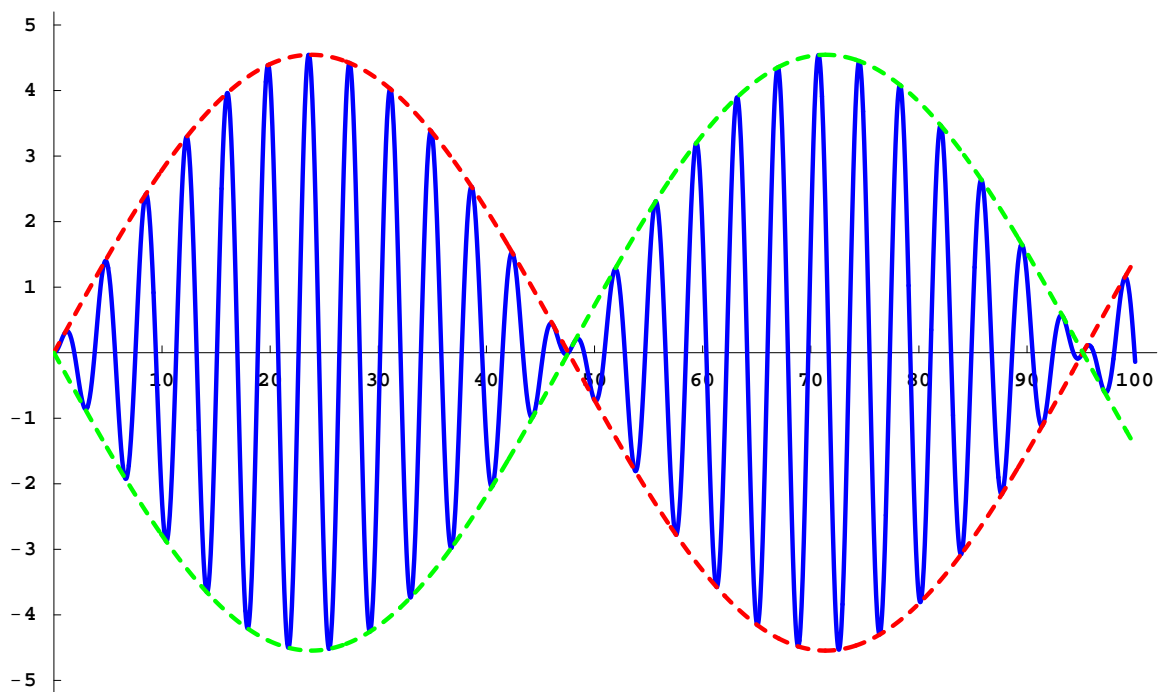
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3 - \omega^2} (\sin \alpha t) (\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where  $\omega = 1.6$ .



What happens if  $\omega = \sqrt{3}$ ?

**Example.**

$$\frac{d^2y}{dt^2} + 3y = \cos \sqrt{3}t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}t}.$$

What guess should we use?

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Here is the graph of  $y_p(t)$ .

