1. (20 points) Find the arc length of the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\,\mathbf{k}$$

from the point $(1, 2, 0)$ to the point $(e^2, 2e, 1)$.

$$\text{arc length} = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

$t = 1$ corresponds to $(1, 2, 0)$
$t = e$ corresponds to $(e^2, 2e, 1)$

$$\vec{r}\,'(t) = 2t\,\vec{i} + 2\vec{j} + \frac{1}{t}\vec{k}$$

$$\text{arc length} = \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}}\,dt$$

$$= \int_1^e \sqrt{\left(2t + \frac{1}{t}\right)^2}\,dt$$

$$= \int_1^e \left(2t + \frac{1}{t}\right)dt$$

$$= \Big[t^2 + \ln t\Big]_1^e$$

$$= (e^2 + 1) - 1 = e^2$$

2. (20 points) Let $f(x, y) = e^{2x}\sin(3x + y)$.

(a) Calculate $\frac{\partial f}{\partial x}$.

$$\frac{\partial f}{\partial x} = 2e^{2x}\sin(3x+y) + e^{2x}\cos(3x+y)(3)$$

$$= e^{2x}\left(2\sin(3x+y) + 3\cos(3x+y)\right)$$

(b) Calculate the gradient vector $\nabla f(0, \pi/4)$

$$\frac{\partial f}{\partial y} = e^{2x}\cos(3x+y)$$

$$\nabla f\left(0, \tfrac{\pi}{4}\right) = \left(2\frac{\sqrt{2}}{2} + 3\frac{\sqrt{2}}{2}\right)\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$$

$$= \left(\frac{5}{2}\sqrt{2}\right)\vec{i} + \left(\frac{\sqrt{2}}{2}\right)\vec{j}$$

(c) Calculate the directional derivative of f in the direction $\mathbf{v}$ of most rapid increase for $f(x, y)$ at the point $(0, \pi/4)$. Assume that $||\mathbf{v}|| = 1$.

direction of most rapid increase is direction of the gradient. Also, directional derivative $= \|\nabla f(0, \tfrac{\pi}{4})\| \cos\theta$ where θ = angle between $\vec{v}$ and ∇f.

Here $\theta = 0 \Rightarrow$ dir derivative $= \|\nabla f\|$

$$= \sqrt{\left(\frac{50}{4}\right) + \left(\frac{2}{4}\right)} = \sqrt{13}.$$

3. (20 points) Derive the second-order Taylor polynomial of the function

$$f(x, y) = \log(x + 2y)$$

at the point $(2, 1)$. Express your answer in terms of x and y rather than in terms of h_1 and h_2. Note: I am using the book's notation of log to represent the **natural logarithm**.

$$\frac{\partial f}{\partial x} = \frac{1}{x+2y} \qquad \frac{\partial f}{\partial y} = \frac{2}{x+2y}$$

$$\frac{\partial^2 f}{\partial x^2} = -1(x+2y)^{-2} \qquad \frac{\partial^2 f}{\partial y^2} = -2(x+2y)^{-2}(2)$$

$$= -4(x+2y)^{-2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2(x+2y)^{-2}.$$

At $(2,1)$, $x+2y = 4$. $f(2,1) = \log 4$

$$p(x,y) = \log 4 + \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix} +$$

$$\frac{1}{2} \begin{bmatrix} x-2 & y-1 \end{bmatrix} \begin{bmatrix} -\frac{1}{16} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix}$$

$$p(x,y) = \log 4 + \frac{1}{4}(x-2) + \frac{1}{2}(y-1)$$

$$- \frac{1}{32}(x-2)^2 - \frac{1}{8}(x-2)(y-1) - \frac{1}{8}(y-1)^2$$

4. (20 points) Let P be the plane $x - 2y - 2z = 1$. Find equations for the two planes that are parallel to P and five units away from P.

$\vec{n}$ = normal vector $= \vec{i} - 2\vec{j} - 2\vec{k}$

$\|\vec{n}\| = \sqrt{1+4+4} = 3$

Point on P is $(1,0,0)$

Two points – one for each plane:

$(1,0,0) + \frac{5}{3}(1,-2,-2) = \left(\frac{8}{3}, -\frac{10}{3}, -\frac{10}{3}\right)$

$(1,0,0) - \frac{5}{3}(1,-2,-2) = \left(-\frac{2}{3}, \frac{10}{3}, \frac{10}{3}\right)$

Equation for 1 plane:

$(\vec{i} - 2\vec{j} - 2\vec{k}) \cdot \left(\left(x - \frac{8}{3}\right)\vec{i} + \left(y + \frac{10}{3}\right)\vec{j} + \left(z + \frac{10}{3}\right)\vec{k}\right) = 0$

$x - \frac{8}{3} - 2\left(y + \frac{10}{3}\right) - 2\left(z + \frac{10}{3}\right) = 0$

$x - 2y - 2z = \frac{48}{3} = 16$

Equation for the other plane:

$\left(x + \frac{2}{3}\right) - 2\left(y - \frac{10}{3}\right) - 2\left(z - \frac{10}{3}\right) = 0$

$x - 2y - 2z = -\frac{42}{3} = -14$

5. (20 points) Parameterize the line of intersection of the plane tangent to $z = e^x \sin y$ at $(0, \pi/2, 1)$ and the plane tangent to $z = x^2 + 2y^2$ at $(1, 2, 9)$ using a vector-valued function $\mathbf{L}(t)$.

$\vec{n} = \left(\frac{\partial z}{\partial x}\right)\vec{i} + \left(\frac{\partial z}{\partial y}\right)\vec{j} - \vec{k}$

For $z = e^x \sin y$ at $(0, \frac{\pi}{2}, 1)$, we get

$\vec{n}_1 = \vec{i} - \vec{k}$.

For $z = x^2 + 2y^2$ at $(1,2,9)$, we get

$\vec{n}_2 = 2\vec{i} + 8\vec{j} - \vec{k}$.

line of intersection is normal to both $\vec{n}_1$ and $\vec{n}_2$

direction vector $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 2 & 8 & -1 \end{bmatrix}$

$= 8\vec{i} - \vec{j} + 8\vec{k}$.

One tangent plane is $x - (z-1) = 0$

$x - z = -1$

other tangent plane is $2(x-1) + 8(y-2) - (z-9) = 0$

$2x + 8y - z = 9$

Points on the line of intersection satisfy $x = z - 1$ and $x + 8y = 10$, e.g., $(2,1,3)$

$\vec{L}(t) = (2+8t)\vec{i} + (1-t)\vec{j} + (3+8t)\vec{k}$.