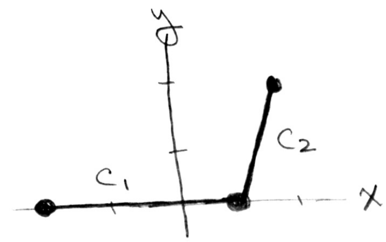
1. (20 points) Let $C$ be the curve in the $xy$-plane that starts at $(-2,0)$, goes along the $x$-axis until it reaches the point $(1,0)$, and then follows the line segment from $(1,0)$ to the point $(2,2)$. Calculate the line integral

$$\int_C 2x\,dx + xy\,dy.$$




$$\int_C = \int_{C_1} + \int_{C_2}$$

Along $C_1$, $y=0$ and

$$\int_{C_1} 2x\,dx + xy\,dy = \int_{-2}^{1} 2x\,dx = \left[x^2\right]_{-2}^{1}$$

$$= 1 - 4 = -3.$$

Along $C_2$, $y = 2x-2$ and $1 \le x \le 2$.

$$\int_{C_2} 2x\,dx + xy\,dy =$$

$$\int_1^2 2x\,dx + x(2x-2)(2)\,dx =$$

$$\int_1^2 (2x + 4x^2 - 4x)\,dx =$$

$$\int_1^2 (4x^2 - 2x)\,dx = \left[\frac{4x^3}{3} - x^2\right]_1^2$$

$$= \left(\frac{32}{3} - 4\right) - \left(\frac{4}{3} - 1\right)$$

$$= \frac{32}{3} - \frac{12}{3} - \frac{1}{3} = \frac{19}{3}$$

Combining $\int_{C_1}$ and $\int_{C_2}$ we get $\int_C = \frac{10}{3}$

2. (20 points) Evaluate the triple integral

$$\iiint_E 2z\, dV,$$

where $E$ is the solid that lies inside the cylinder $x^2 + y^2 = 3$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

Use cylindrical coordinates.

$$\iiint_E 2z\, dV =$$

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{2r} (2z)\, r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^{\sqrt{3}} r(4r^2)\, dr\, d\theta =$$

$$\int_0^{2\pi} \left[r^4\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi} 9\, d\theta$$

$$= 18\pi$$

3. (20 points) Find the surface area of the portion of the circular paraboloid $z = 16 - x^2 - y^2$ that lies between the planes $z = 4$ and $z = 12$.

$z = f(x,y) = 16 - x^2 - y^2$

$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$ ← area in xy-plane

surface area ↑

$= \sqrt{4x^2 + 4y^2 + 1}\ dA$

$z = 4 \Rightarrow 4 = 16 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 12$

$z = 12 \Rightarrow 12 = 16 - x^2 - y^2$

$x^2 + y^2 = 4$

Region of integration in xy-plane satisfies $2 \le r \le 2\sqrt{3}$.

surface area $= \int_0^{2\pi} \int_2^{2\sqrt{3}} \sqrt{4r^2+1}\ r\,dr\,d\theta$

$u = 4r^2 + 1 \Rightarrow r = 2 \Rightarrow u = 17$

$du = 8r\,dr \qquad r = 2\sqrt{3} \Rightarrow u = 49$

$\int_2^{2\sqrt{3}} \sqrt{4r^2+1}\ r\,dr = \left[\frac{1}{8}\,\frac{u^{3/2}}{(3/2)}\right]_{17}^{49} = \frac{1}{12}\left[343 - 17\sqrt{17}\right]$

area $= \frac{\pi}{6}\left[343 - 17\sqrt{17}\right]$

4. (20 points) Suppose $T$ is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and $S$ is the surface that bounds $T$. Use the Divergence Theorem to calculate the flux across $S$ in the outward normal direction for the vector field

$$\mathbf{F}(x,y,z) = x^2\mathbf{i} + (x-z)\mathbf{j} + (y^2-x)\mathbf{k}.$$

$\operatorname{div}\vec{F} = 2x$

$$\iiint_T (\operatorname{div}\vec{F})\,dV = \iiint_T 2x\,dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (2x)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (2x)(1-x-y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (2x - 2x^2 - 2xy)\,dy\,dx$$

$$= \int_0^1 2x(1-x) - 2x^2(1-x) - x(1-x)^2\,dx$$

$$= \int_0^1 2x - 2x^2 - 2x^2 + 2x^3 - x + 2x^2 - x^3\,dx$$

$$= \int_0^1 x^3 - 2x^2 + x\,dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{2}{3} + \frac{1}{2}$$

$$= \frac{1}{12}$$

5. (20 points)

(a) Calculate the line integral

$$\int_{C_1} (5y+2)\,dx + 3x\,dy$$

where $C_1$ is the line segment in the $xy$-plane from $(1,0)$ to $(-1,0)$.

$x = 1 - t$

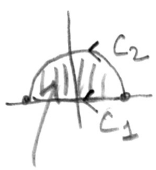
$dx = -dt$

$$\int_{C_1} = \int_0^2 (2)(-1)\,dt$$

$$= -4.$$

(b) Using Green's Theorem and your result in Part (a), calculate the line integral

$$\int_{C_2} (5y+2)\,dx + 3x\,dy$$




H

where $C_2$ is the positively-oriented curve that runs along the unit circle $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$ in the upper half-plane.

$$\int_{C_2 - C_1} (5y+2)\,dx + 3x\,dy = \iint_H (3-5)\,dA$$

$$= -2 \text{ area } H$$

$$= -2\left(\frac{\pi}{2}\right) = -\pi$$

$$\Rightarrow \int_{C_2} (5y+2)\,dx + 3x\,dy = -\pi - 4$$