Definition of Limit
Let $f : A \to \mathbb{R}^m$ where $A \subset \mathbb{R}^n$ is open, and suppose $\mathbf{x}_0 \in (A \cup \partial A)$ and $\mathbf{b} \in \mathbb{R}^m$.

Definition

1. Let N be a neighborhood of $\mathbf{b}$. Then $f(\mathbf{x})$ is eventually in N as $\mathbf{x} \to \mathbf{x}_0$ ("$\mathbf{x}$ approaches $\mathbf{x}_0$") if there exists a neighborhood U of $\mathbf{x}_0$ such that
$$f((U \cap A) - \{\mathbf{x}_0\}) \subset N.$$
2. We say that $f(\mathbf{x}) \to \mathbf{b}$ ("$f(\mathbf{x})$ approaches $\mathbf{b}$") as $\mathbf{x} \to \mathbf{x}_0$ if, given any neighborhood N of $\mathbf{b}$, $f(\mathbf{x})$ is eventually in N.

The notation
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$
is equivalent to part 2 of the definition.
If $f(\mathbf{x})$ does not approach any $\mathbf{b} \in \mathbb{R}^m$ (as in part 2 above), then we say that
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$$
does not exist.

Properties of Limits

1. If $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x} \to \mathbf{x}_0} cf(\mathbf{x}) = c\mathbf{b}$ for any scalar c.
2. Limit of the sum is the sum of the limits.
3. If f and g are both scalar valued, then the limit of the product is the product of the limits.
4. If f is scalar valued and nonzero in a neighborhood of $\mathbf{x}_0$ and the limit of f is nonzero, then the limit of $1/f$ is the reciprocal of the limit of f.
5. If f is vector valued, then the limit of f is determined by the limits of all of its component functions.

Each one of these facts about limits yields a corresponding result about continuous functions.

Continuity of Compositions
If g is continuous at $\mathbf{x}_0$ and f is continuous at $g(\mathbf{x}_0)$, then the composition $f \circ g$ is continuous at $\mathbf{x}_0$.

Infinite Sequence Characterization of Limits (Exercise 16 on page 164)
An infinite sequence of points $\mathbf{x}_k$ ($k = 1, 2, 3, \ldots$) in $\mathbb{R}^m$ is said to **converge to** $\mathbf{x}_0$ as $k \to \infty$ if, for any $\epsilon > 0$, there is some positive integer K (depending only on ϵ) such that $k \geq K$ implies that $||\mathbf{x}_k - \mathbf{x}_0|| < \epsilon$. We often write $\mathbf{x}_k \to \mathbf{x}_0$ as $k \to \infty$.

Informally, this definition is equivalent to the assertion that the sequence $\mathbf{x}_k$ is eventually in every neighborhood of $\mathbf{x}_0$.

Limits via Sequences
Let $\mathbf{x}_0$ be a point in A and $f : A \to \mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$

if and only if $f(\mathbf{x}_k) \to \mathbf{b}$ for every sequence $\mathbf{x}_k \to \mathbf{x}_0$ in A.

In other words, we have the limit

$$f(\mathbf{x}) \to \mathbf{b} \text{ as } \mathbf{x} \to \mathbf{x}_0$$

if and only if the sequence of images

$$f(\mathbf{x}_k) \to \mathbf{b}$$

for *every* sequence $\mathbf{x}_k \to \mathbf{x}_0$ in A.

Comments:

1. Note that this characterization of limits via sequences uses two sequences—the sequence $\mathbf{x}_k \to \mathbf{x}_0$ and the sequence of the corresponding images $f(\mathbf{x}_k)$.

2. This characterization is not always so easy to use because you have to show that $f(\mathbf{x}_k) \to \mathbf{b}$ for *every* sequence $\mathbf{x}_k \to \mathbf{x}_0$, and there are infinitely many such sequences.

3. However, this characterization is very useful if you want to show that a limit does not exist. If you find a sequence approaching $\mathbf{x}_0$ whose image sequence does not approach any particular $\mathbf{b}$, then the limit does not exist. Also, if you find two different sequences approaching $\mathbf{x}_0$ and their image sequences approach different limits, then the limit does not exist.