Consider the vector field

$$\mathbf{F}(x, y, z) = (x^2 + y \sin z)\mathbf{i} + (x \sin z + 2y)\mathbf{j} + (xy \cos z)\mathbf{k}.$$

1. Compute the curl of $\mathbf{F}$.
2. Compute a potential function for $\mathbf{F}$.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y\sin z & x\sin z + 2y & xy\cos z \end{bmatrix}$$

$$= (\sin z - \sin z)\vec{i} - (y\cos z - y\cos z)\vec{j}$$

$$+ (\sin z - \sin z)\vec{k}$$

$$= \vec{0}. \Rightarrow \vec{F} = \nabla f$$

$$f(x,y,z) = \int (xy\cos z)\,dz + \varphi(x,y)$$

$$= xy\sin z + \varphi(x,y)$$

$$\frac{\partial f}{\partial x} = x^2 + y\sin z \Rightarrow \frac{\partial \varphi}{\partial x} = x^2$$

$$\frac{\partial f}{\partial y} = x\sin z + 2y \Rightarrow \frac{\partial \varphi}{\partial y} = 2y$$

$$\varphi(x,y) = \int 2y\,dy + \psi(x)$$

$$= y^2 + \psi(x)$$

arbitrary constant

$$\frac{\partial \varphi}{\partial x} = x^2 \Rightarrow \frac{d\psi}{dx} = x^2 \Rightarrow \psi(x) = \frac{x^3}{3} + C$$

$$f(x,y,z) = xy\sin z + y^2 + \frac{x^3}{3} + C$$