Consider the vector field

$$\mathbf{F}(x, y, z) = (x^3 + y \sin z)\mathbf{i} + (y^3 + z \sin x)\mathbf{j} + (3z)\mathbf{k}.$$

Use the Divergence Theorem (Gauss' Theorem) to calculate the flux of $\mathbf{F}$ across the surface S, where S is the boundary of the solid that is bounded by the hemispheres

$$z = \sqrt{4 - x^2 - y^2}$$

and

$$z = \sqrt{1 - x^2 - y^2}$$

and the plane $z = 0$.

$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3$

R = region bdd by S

$$\text{flux} = \iiint_R (\operatorname{div} \vec{F})\, dV$$

$$= \iiint_R 3x^2 + 3y^2 + 3 \, dV$$

$$= 3 \iiint_R x^2 + y^2 + 1 \, dV$$

use spherical coordinates to describe $R = \left\{ (\rho, \phi, \theta) \;\middle|\; \begin{array}{l} 1 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \end{array} \right\}$

$$\text{flux} = 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 (\rho^2 \sin^2\phi + 1)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 (\rho^4 \sin^3\phi + \rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\pi/2} \left(\frac{31}{5} \sin^3\phi + \frac{7}{3} \sin\phi \right) d\phi\, d\theta$$

Note that $\sin^3\phi = (\sin^2\phi)(\sin\phi)$

$$= (1-\cos^2\phi)(\sin\phi)$$

and $\int \sin^3\phi \, d\phi = -\cos\phi + \frac{\cos^3\phi}{3} + C$.

$$\text{flux} = 3\int_0^{2\pi} \frac{31}{5}\left[\cos\phi - \frac{\cos^3\phi}{3}\right]_{\pi/2}^{0} + \frac{7}{3}\left[\cos\phi\right]_{\pi/2}^{0} \, d\theta$$

$$= 3\int_0^{2\pi} \left(\frac{31}{5}\right)\left(\frac{2}{3}\right) + \left(\frac{7}{3}\right)(1) \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{62}{5} + 7\right) d\theta$$

$$= (2\pi)\left(\frac{97}{5}\right) = \frac{194}{5}\pi$$