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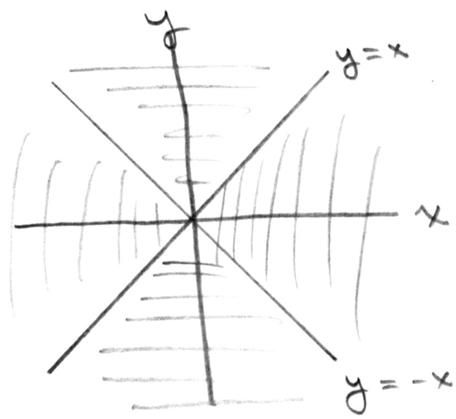
#1) (a) Yes, p is continuous on all of $\mathbb{R}^2$.
To justify this assertion, note that $\mathbb{R}^2$ can be subdivided into 4 regions



Consider, for example, the open region

$$R_1 = \{(x,y) \mid x>0 \text{ and } -x<y<x\}.$$

On R_1, $p(x,y) = x$. Using Example 4 in Section 2.2, we see that p is continuous on R_1.

There are three other open regions that can be discussed similarly.

On $R_2 = \{(x,y) \mid y > 0 \text{ and } -y < x < y\}$,

we have $p(x,y) = y$.

On $R_3 = \{(x,y) \mid x < 0 \text{ and } x < y < -x\}$,

we have $p(x,y) = -x$.

On $R_4 = \{(x,y) \mid y < 0 \text{ and } y < x < -y\}$,

we have $p(x,y) = -y$.

We know ~~that~~ p is continuous on these regions by Example 4 in Section 2.2.

These four regions are bounded by the two lines $y = x$ and $y = -x$.

Suppose (x_0, y_0) satisfies $y = x$ and $x > 0$.

Then $(x_0, y_0) = (x_0, x_0)$ and $p(x_0, y_0) = x_0$.

If a sequence of points $(x_n, y_n) \to (x_0, y_0)$, then $x_n \to x_0$. If $(x_n, y_n) \in R_1$, then

then $p(x_n, y_n) \to x_0 = p(x_0, y_0)$.

If $(x_n, y_n) \in R_2$, then $p(x_n, y_n) \to y_0$,

but $y_0 = x_0$.

Continuity along the other three half-lines is established in a similar fashion.

(b) p is C^1 on $R_1 \cup R_2 \cup R_3 \cup R_4$.

This follows using the same logic as in (a). For example, on R_1, $p(x,y) = x$, which is certainly C^1.

p is not differentiable on the two lines $y = x$ and $y = -x$. To see why, let (x_0, y_0) be a point on the line $y = x$.

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Then $\frac{\partial p}{\partial x}(x_0, x_0) = \lim_{h \to 0} \frac{p(x_0+h, x_0) - p(x_0, x_0)}{h}$

If $h > 0$, we have

$$\frac{p(x_0+h, x_0) - p(x_0, x_0)}{h} = \frac{(x_0+h) - x_0}{h}$$

$$= 1.$$

If $h < 0$, we have

$$\frac{p(x_0+h, x_0) - p(x_0, x_0)}{h} = \frac{x_0 - x_0}{h}$$

$$= 0$$

Therefore, $\lim_{h \to 0} \frac{p(x_0+h, x_0) - p(x_0, x_0)}{h}$ does not exist. Since $\frac{\partial p}{\partial x}(x_0, x_0)$ does not exist, p is not differentiable at (x_0, x_0).

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#2) By definition

$$\frac{\partial f}{\partial y}(b,a) = \lim_{h \to 0} \frac{f(b,a+h) - f(b,a)}{h}$$

But we know that $f(x,y) = f(y,x)$ for all $(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial y}(b,a) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \frac{\partial f}{\partial x}(a,b).$$

#3) Let $\vec{x}_0 \in \mathbb{R}^n$. We must show that

$$\lim_{\vec{x} \to \vec{x}_0} g(\vec{x}) = g(\vec{x}_0).$$

This is equivalent to

$$\lim_{\vec{x} \to \vec{x}_0} \left(g(\vec{x}) - g(\vec{x}_0) \right) = 0.$$

Consider

$$\begin{aligned} g(\vec{x}) - g(\vec{x}_0) &= \vec{x} \cdot \vec{v} - \vec{x}_0 \cdot \vec{v} \\ &= (\vec{x} - \vec{x}_0) \cdot \vec{v} \end{aligned}$$

By the Cauchy-Schwarz inequality, we know that

$$|g(\vec{x}) - g(\vec{x}_0)| \leq \|\vec{x} - \vec{x}_0\| \, \|\vec{v}\|.$$

Since $\|\vec{x} - \vec{x}_0\| \to 0$ as $\vec{x} \to \vec{x}_0$ and $\vec{v}$ is fixed, we see that

$$\lim_{\vec{x} \to \vec{x}_0} |g(\vec{x}) - g(\vec{x}_0)| = 0.$$

#4) Let (x_0, y_0, z_0) be the point of intersection and let t_0 be the associated t value. The vector from $(1,3,-2)$ to the point of intersection is

$$\vec{v} = (2t_0)\vec{i} + (-t_0 - 2)\vec{j} + (4 - 3t_0)\vec{k}$$

and the direction vector of the given line is

$$\vec{w} = 2\vec{i} - \vec{j} - 3\vec{k}.$$

We want $\vec{v} \cdot \vec{w} = 0 \Rightarrow$

$$4t_0 + t_0 + 2 - 12 + 9t_0 = 0$$

$$14t_0 = 10$$

$$t_0 = \frac{5}{7}$$

Therefore, $(x_0, y_0, z_0) = \left(\frac{17}{7}, \frac{2}{7}, -\frac{1}{7}\right)$

and $\vec{v} = \frac{10}{7}\vec{i} - \frac{19}{7}\vec{j} + \frac{13}{7}\vec{k}$

Using the point $(1,3,-2)$ and the direction vector

$$\vec{v} = \frac{10}{7}\vec{i} - \frac{19}{7}\vec{j} + \frac{13}{7}\vec{k},$$

we obtain the equation

$$x = 1 + \frac{10}{7}t$$

$$y = 3 - \frac{19}{7}t$$

$$z = -2 + \frac{13}{7}t.$$

If you don't like fractions, use $7\vec{v}$ as the direction vector. We obtain

$$x = 1 + 10t$$
$$y = 3 - 19t$$
$$z = -2 + 13t.$$

#5)

(a) Using polar coordinates, $r^2 = x^2 + y^2$ so

$$f(x,y) = \frac{\sin(r^2)}{r^2}.$$

As $(x,y) \to (0,0)$, $r \to 0$ and therefore

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r \to 0} \frac{\sin(r^2)}{r^2}$$

$$= 1.$$

If we define $f(0,0) = 1$, then $f(x,y)$ is continuous at $(0,0)$.

It is also continuous for all $(x_0, y_0) \neq (0,0)$ because we can apply the continuity theorems to the quotient

$$\frac{\sin(x^2+y^2)}{x^2+y^2}$$

(b) We must compute the partials

$\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sin(h^2)}{h^2} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h^2) - h^2}{h^3}$$

$$= 0$$

(This limit can be calculated various ways, e.g., l'Hôpital's Rule.)

Since the definition of f is symmetric in x and y, we also have $\frac{\partial f}{\partial y}(0,0) = 0$.

Since both partials vanish, we have

$$R_2(h_1,h_2) = f(h_1,h_2) - f(0,0)$$

$$= \frac{\sin(h_1^2+h_2^2)}{h_1^2+h_2^2} - 1 .$$

To determine differentiability, we must show that

$$\lim_{\vec{h}\to\vec{0}} \frac{R_2(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} \stackrel{?}{=} 0$$

In other words, we want

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\sin(h_1^2+h_2^2) - (h_1^2+h_2^2)}{(h_1^2+h_2^2)^{3/2}} \stackrel{?}{=} 0$$

Let $u = h_1^2 + h_2^2$. As $(h_1,h_2)\to(0,0)$, $u\to 0^+$. We must compute

$$\lim_{u\to 0^+} \frac{(\sin u) - u}{u^{3/2}} .$$

Again there are various ways to compute this limit. We get

$$\lim_{u \to 0^+} \frac{(\sin u) - u}{u^{3/2}} = 0.$$

(c) For $(x_0, y_0) \neq (0,0)$, we can prove that f is differentiable by proving that it is C^1.

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)\cos(x^2+y^2)(2x) - (2x)\sin(x^2+y^2)}{(x^2+y^2)^2}$$

Using our continuity theorems, we know that this function is continuous on

$$\mathbb{R}^2 - \{(0,0)\}.$$

The same assertion applies to $\frac{\partial f}{\partial y}$, and therefore $f(x,y)$ is C^1 on $\mathbb{R}^2 - \{(0,0)\}$.

#6) (a) The function is differentiable on $\mathbb{R}^2 - \{(0,0)\}$ because it is C^1 on this open set. To see why, compute

$$\frac{\partial f}{\partial x} = \frac{(x^4+y^2)^2(2xy) - (x^2y)(4x^3)}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^4+y^2)^2(x^2) - (x^2y)(2y)}{(x^4+y^2)^2}$$

By our continuity theorems, both of these functions are continuous on $\mathbb{R}^2 - \{(0,0)\}$.

(b) Let $\vec{v} = (\cos\theta, \sin\theta)$. The directional derivative of f at $(0,0)$ in the direction of $\vec{v}$ is

$$\lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} =$$

$$\lim_{h\to 0} \frac{(h^2\cos^2\theta)(h\sin\theta)}{h(h^4\cos^4\theta + h^2\sin^2\theta)} =$$

$$\lim_{h \to 0} \frac{(\cos^2\theta)(\sin\theta)}{h^2\cos^4\theta + \sin^2\theta}$$

If $\sin\theta \neq 0$ ($\theta \neq k\pi$ for $k \in \mathbb{Z}$), then this limit is

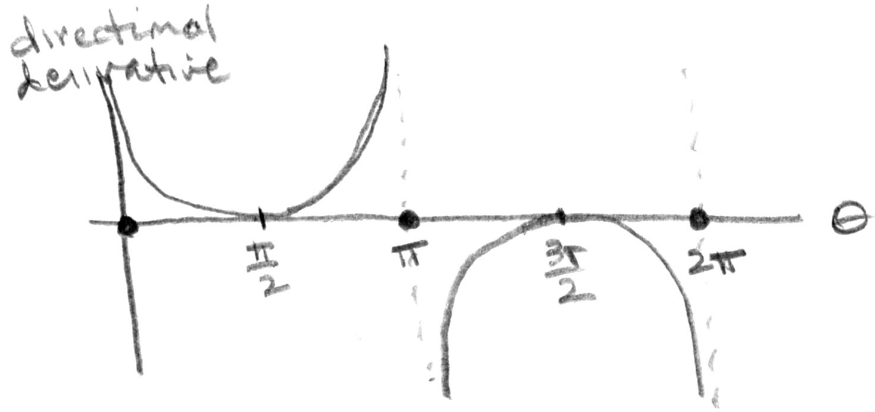
$$\frac{(\cos^2\theta)(\sin\theta)}{\sin^2\theta} = \frac{\cos^2\theta}{\sin\theta}$$

If $\theta = 0$, we must calculate

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{(h^2)(0)}{h^4 + 0} = 0$$

We get the same value for $\theta = \pi$.

(c) As $\vec{v}$ sweeps out the unit circle, our directional derivatives vary as a function of θ. The graph is



(d) Let $y = \alpha x^2$. Then

$$f(x,y) = \frac{x^2(\alpha x^2)}{x^4 + \alpha^2 x^4}$$

$$= \frac{\alpha}{1+\alpha^2}$$

In other words, the punctured parabola

$$y = \alpha x^2, \quad x \neq 0,$$

is the level set of level $\frac{\alpha}{1+\alpha^2}$.

So if $(x,y) \to 0$ along one of these parabolas, then the limiting value is $\frac{\alpha}{1+\alpha^2}$. f cannot be continuous at $(0,0)$.