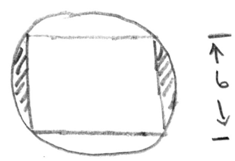
1. (20 points) The following volume problem appeared in Marilyn Vos Savant's April 20, 1996 column:

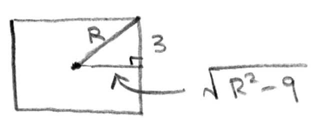
> Start with a solid sphere of any radius greater than 6 inches. Bore a cylindrical hole through the center of the sphere so that what remains is exactly 6 inches high. (The center line of the cylinder should correspond to a diameter of the sphere.) What is the volume of the solid that remains after the cylindrical hole is removed?

Calculate this volume. Does it depend on the radius R of the original sphere?

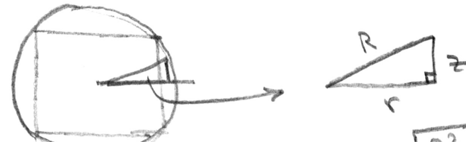
Let R be the radius of the sphere. We have



In terms of cylindrical coordinates, we can descibe the remaining regin by



and



$z = \sqrt{R^2 - r^2}$

In other words, the regin is given by

$$\left\{ (r, \theta, z) \mid 0 \le \theta \le 2\pi,\ \sqrt{R^2-9} \le r \le R,\ -\sqrt{R^2-r^2} \le z \le \sqrt{R^2-r^2} \right\}$$

Therefore,

$$\text{volume} = \int_0^{2\pi} \int_{\sqrt{R^2-9}}^{R} \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r \, dz\, dr\, d\theta.$$

$$\text{volume} = \int_0^{2\pi} \int_{\sqrt{R^2-9}}^{R} 2\sqrt{R^2-r^2}\; r\,dr\,d\theta$$

If $u = R^2 - r^2$ then

$$du = -2r\,dr$$

$$r = R \Rightarrow u = 0$$

$$r = \sqrt{R^2-9} \Rightarrow u = 9.$$

Then

$$\text{volume} = \int_0^{2\pi} \int_0^9 u^{\frac{1}{2}}\,du\,d\theta.$$

Note that this integral does not depend on R. We get

$$\text{volume} = \int_0^{2\pi} \left[\frac{2}{3}\,u^{3/2}\right]_0^9 d\theta$$

$$= \int_0^{2\pi} 18\,d\theta = 36\pi.$$

2. (20 points) Suppose the D is an elementary region in $\mathbb{R}^3$ that is symmetric with respect to the yz-plane. In other words, a point (x, y, z) is in D if and only if the point $(-x, y, z)$ is in D. Using the Change of Variables Theorem, show that

$$\iiint_D f(x, y, z)\, dV = 0$$

if $f(x, y, z)$ is an odd function in x. (A function $f : \mathbb{R}^3 \to \mathbb{R}$ is odd in x if $f(-x, y, z) = -f(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$.)

Subdivide D into two regions D_1 and D_2 such that

$$D_1 = \{(x,y,z) \mid (x,y,z) \in D \text{ and } x \geq 0\}$$

$$D_2 = \{(x,y,z) \mid (x,y,z) \in D \text{ and } x \leq 0\}.$$

Then $\iiint_D f\, dV = \iiint_{D_1} f\, dV + \iiint_{D_2} f\, dV$.

Consider the mapping $T: D_1 \to D_2$ given by $T(x,y,z) = (-x, y, z)$

The Jacobian of T is

$$\left| \det \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 1$$

By the Change of Variables Therem, we have

$$\iiint_{D_2} f(x,y,z)\, dV = \iiint_{D_1} f \circ T(x,y,z)\, |-1|\, dV$$

$$= \iiint_{D_1} f(-x,y,z)\, dV$$

$$= -1 \iiint_{D_1} f(x,y,z)\, dV$$

Since $\displaystyle\iiint_{D_2} f\, dV = -\iiint_{D_1} f\, dV$,

we see that $\displaystyle\iiint_{D} f\, dV = 0$.

3. (20 points) The method of least squares approximation is described on page 246 of our text. In this problem, we use notation that is discussed there.

   (a) Consider the three data points $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 3)$, and $(x_3, y_3) = (3, 2)$. Find the equation of the best straight-line fit according to the method of least squares. Calculate the equation for the line directly. (In other words, don't use the result of Exercise 32 on page 247.) Give a rigorous justification of why this line minimizes s.

   (b) Solve Exercise 32 on page 247, and use the result to give explicit formulas for m and b.

   (c) Solve Exercise 34 on page 247.

   (d) Use the method of least squares to find the line that best fits the points $(0, 2)$, $(1, 4)$, $(1, 3)$, $(2, 6)$, and $(3, 6)$. Plot the points and the line.

(a) We want to minimize

$$s(m,b) = \underset{\substack{\uparrow \\ \text{data point} \\ (1,1)}}{(m+b+1)^2} + \underset{\substack{\uparrow \\ \text{data point} \\ (2,3)}}{(2m+b-3)^2} + \underset{\substack{\uparrow \\ \text{data point} \\ (3,2)}}{(3m+b-2)^2}.$$

After some algebra, we obtain

$$S = 14m^2 + 12mb - 26m + 3b^2 - 12b + 14,$$

and

$$\frac{\partial S}{\partial m} = 28m + 12b - 26$$

$$\frac{\partial S}{\partial b} = 12m + 6b - 12 .$$

To determine the critical points, we solve

$$\begin{cases} \dfrac{\partial S}{\partial m} = 0 \\ \dfrac{\partial S}{\partial b} = 0 . \end{cases}$$

We obtain one critical point,

$$(m,b) = \left(\tfrac{1}{2}, 1\right)$$

Applying the second partials test, we compute

$$\frac{\partial^2 s}{\partial m^2} = 28 \qquad \frac{\partial^2 s}{\partial b^2} = 6$$

$$\frac{\partial^2 s}{\partial m \partial b} = 12 .$$

Note that $\frac{\partial^2 s}{\partial m^2} > 0$ and

$$\left(\frac{\partial^2 s}{\partial m^2}\right)\left(\frac{\partial^2 s}{\partial b^2}\right) - \left(\frac{\partial^2 s}{\partial m \partial b}\right)^2 > 0 ,$$

and we have a local minimum at the critical point.

There <u>are</u> functions $\mathbb{R}^2 \to \mathbb{R}$ that have exactly one critical point which is a local minimum but not a global minimum! We can show that $s(m, b)$ is not one of these functions using some analytic geometry.

(b) $S(m,b) = \sum (y_i - m x_i - b)^2$

Then $\frac{\partial S}{\partial m} = -2 \sum (y_i - m x_i - b)(x_i)$

$$\frac{\partial S}{\partial b} = -2 \sum (y_i - m x_i - b).$$

The critical points satisfy

$$\sum (x_i y_i - m x_i^2 - b x_i) = 0$$

$$\sum (y_i - m x_i - b) = 0.$$

The second equation implies

$$m \sum x_i + nb = \sum y_i.$$

n terms in sum

The first equation implies

$$m \sum x_i^2 + b \sum x_i = \sum x_i y_i.$$

If we let S_x denote Σx_i

S_y denote Σy_i

S_{x^2} denote Σx_i^2

S_{xy} denote $\Sigma x_i y_i$,

then we have the system of two equations in two unknowns (m and b)

$$(S_x)\, m + n\, b = S_y$$

$$(S_{x^2})\, m + (S_x)\, b = S_{xy}.$$

We can write this system as a matrix equation

$$\begin{bmatrix} S_x & n \\ S_{x^2} & S_x \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} S_y \\ S_{xy} \end{bmatrix}.$$

Solving we obtain

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} S_x & n \\ S_{x^2} & S_x \end{bmatrix}^{-1}\begin{bmatrix} S_y \\ S_{xy} \end{bmatrix} = \frac{1}{(S_x)^2 - nS_{x^2}}\begin{bmatrix} (S_x)(S_y) - n\,S_{xy} \\ (S_x)(S_{xy}) - (S_{x^2})(S_y) \end{bmatrix}$$

(c) Using the formulas for $\frac{\partial s}{\partial m}$ and $\frac{\partial s}{\partial b}$ in part (b), we have

$$\frac{\partial^2 s}{\partial m^2} = 2 \sum x_i^2$$

$$\frac{\partial^2 s}{\partial b^2} = 2n$$

$$\frac{\partial^2 s}{\partial m \partial b} = 2 \sum x_i$$

It is clear that $\frac{\partial^2 s}{\partial m^2} > 0$. We need only compute

$$\left(\frac{\partial^2 s}{\partial m^2}\right)\left(\frac{\partial^2 s}{\partial b^2}\right) - \left(\frac{\partial^2 s}{\partial m \partial b}\right)^2 .$$

Both terms have a factor of 4 that we ignore The rest is

$$n \sum x_i^2 - \left(\sum x_i\right)^2 .$$

To see that this expression is positive, we eliminate all terms of the form x_i^2 from the second sum and obtain

$$(n-1)\sum x_i^2 - 2\Big(\sum_{1\le i<j\le n} x_i x_j\Big)$$

Note that the second sum involves $1+2+3+\cdots+n-1$ terms. Summing we see that there are

$$\frac{(n-1)n}{2}$$

terms. For each such term

$$-2x_i x_j,$$

we select one x_i^2 and one x_j^2 from

$$(n-1)\sum x_i^2 .$$

Since there are $(n-1)n$ such terms in this sum, we have enough terms to write

$$(n-1)\sum x_i^2 - 2\Big(\sum_{1\le i<j\le n} x_i x_j\Big) =$$

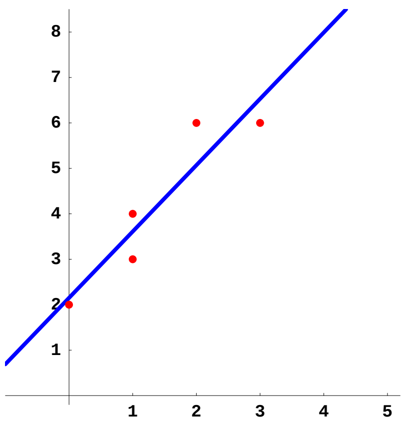
$$\sum_{1\le i<j\le n} (x_i - x_j)^2 .$$

Since this sum is clearly positive, the second derivative test implies that we have a local minimum.

(d) Using the formulas in part (b) we obtain

$$m = \frac{19}{13} \approx 1.46$$

$$b = \frac{28}{13} \approx 2.15 .$$


8
7
6
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4
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1
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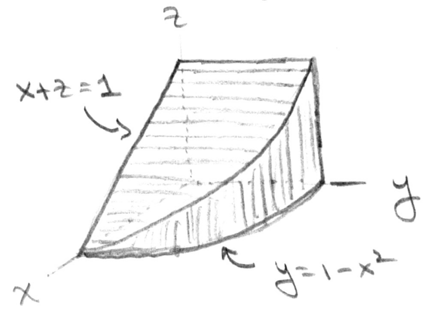
5

4. (20 points) Rewrite the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dz\, dy\, dx$$

as an equivalent iterated integral in the five other orders.

First we sketch the solid region of integration.

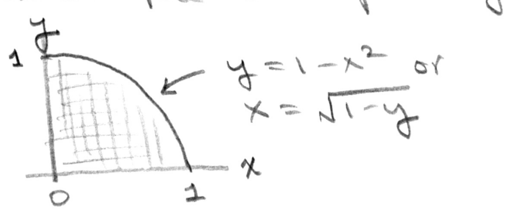


The solid has five "sides." Three of the sides lie in the coordinate planes.

The original integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dz\, dy\, dx$$

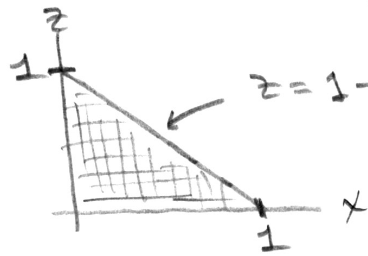
involves projecting the region into the xy-plane. The corresponding side is



If we interchange the order in the xy-plane, we get

$$\int_0^1 \int_0^{\sqrt{1-y}} \int_0^{1-x} f(x,y,z)\, dz\, dx\, dy$$

The projection of the region into the xz-plane is the triangle

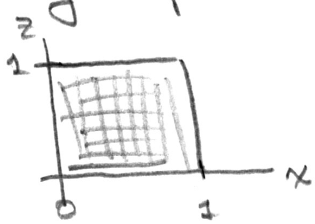


Using this triangle as our "base", we obtain the two integrals

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dy\, dz\, dx \quad \text{and}$$

$$\int_0^1 \int_0^{1-z} \int_0^{1-x^2} f(x,y,z)\, dy\, dx\, dz \ .$$

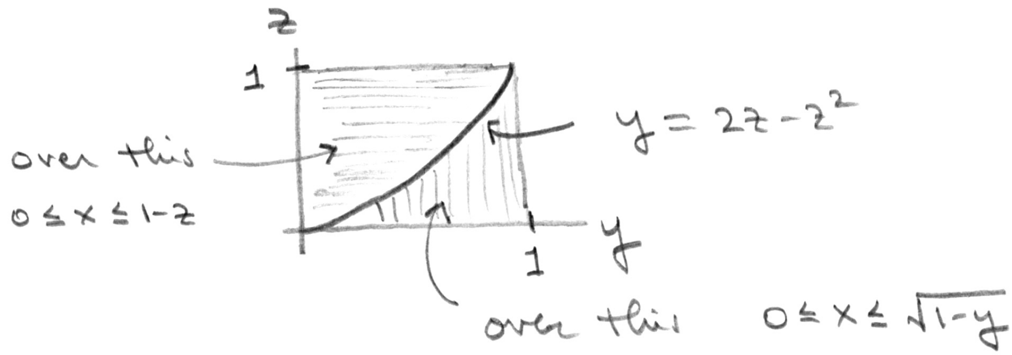
Finally the projection of the region into the yz-plane is a square



Unfortunately we need to subdivide the square into two subregions based on the projection of the boundary curve common to the plane $x+z=1$ and the surface $y=1-x^2$.

Since $x+z=1$ and
$y=1-x^2$

we have $y = 1-(1-z)^2$
$= 2z - z^2$.



The last two orders of integration must be expressed as the sum of two separate integrals:

For $dx\,dy\,dz$, we have

$$\int_0^1 \int_0^{2z-z^2} \int_0^{1-z} f(x,y,z)\,dx\,dy\,dz \;+$$

$$\int_0^1 \int_{2z-z^2}^{1} \int_0^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz$$

For $dx\,dz\,dy$, we must solve for z in the equation $y = 2z - z^2$. We apply the quadratic formula to

$$z^2 - 2z + y = 0,$$

and we obtain $z = 1 \pm \sqrt{1-y}$.

In our case, $z = 1 - \sqrt{1-y}$. The last integral is

$$\int_0^1 \int_0^{1-\sqrt{1-y}} \int_0^{\sqrt{1-y}} f(x,y,z)\, dx\, dz\, dy \; +$$

$$\int_0^1 \int_{1-\sqrt{1-y}}^{1} \int_0^{1-z} f(x,y,z)\, dx\, dz\, dy$$

5. (20 points) Suppose $0 < a < b$. Let

$$\Phi(u, v) = (b \cos u + a \cos v \cos u, b \sin u + a \cos v \sin u, a \sin v).$$

(a) Show that $\Phi(u, v)$ parametrizes the torus that is obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$.

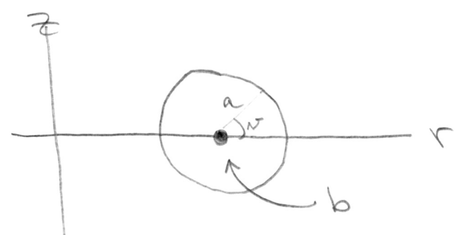
(b) Calculate the surface area of this torus.

(a) Consider the slice of the torus by the half-plane described in cylindrical coordinates as

θ_0 fixed and $r > 0$

This slice is a circle of radius a centered at

$$(r, \theta, z) = (b, \theta_0, 0)$$



We can parametrize this circle using sine and cosine.

$$r = b + a \cos v$$
$$z = a \sin v$$

The v-interval $0 \le v \le 2\pi$ sweeps out this circle exactly once.

Converting this description of the torus to rectangular coordinates yields the parametrization

$$x = r\cos u = (b + a\cos v)\cos u$$
$$y = r\sin u = (b + a\cos v)\sin u$$
$$z = a\sin v$$

The variable u is identical to the coordinate θ in cylindrical coordinates.

(b) $$\Phi_u = \left[-(b + a\cos v)\sin u\right]\vec{i} + \left[(b + a\cos v)\cos u\right]\vec{j}$$

$$\Phi_v = \left[-a\cos u\sin v\right]\vec{i} + \left[-a\sin u\sin v\right]\vec{j} + \left[a\cos v\right]\vec{k}$$

Then

$$\Phi_u \times \Phi_v = [a(b + a\cos v)(\cos u)(\cos v)]\vec{i}$$

$$+ [a(b + a\cos v)(\sin u)(\cos v)]\vec{j}$$

$$+ [a(b + a\cos v)(\sin v)]\vec{k},$$

and

$$\|\Phi_u \times \Phi_v\|^2 = a^2(b + a\cos v)^2.$$

Since $dS = \|\Phi_u \times \Phi_v\|\, dA$ ↖ uv-plane

$$\text{surface area} = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos v)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} (ab + a^2\cos v)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} (ab)\, du\, dv$$

$$= 4\pi^2 ab$$