Here is some background related to Exercise 15 on page 74:

Definition of a Vector Space

A (real) **vector space** V is a set along with two operations—“vector” addition and scalar multipication. (The elements of V are called “vectors” even though they may have little to do with the classical idea of a vector.) The operation of vector addition produces a vector $\mathbf{u} + \mathbf{v} \in V$ from a pair of vectors $\mathbf{u},\ \mathbf{v} \in V$, and the operation of scalar multipication produces a vector $c\mathbf{u} \in V$ from a vector $\mathbf{u} \in V$ and a scalar (real number) c. These two operations must also satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u},\ \mathbf{v} \in V$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$.
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
4. For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for all $\mathbf{u},\ \mathbf{v} \in V$ and all $c \in \mathbb{R}$.
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for all $\mathbf{u} \in V$ and all $c,\ d \in \mathbb{R}$.
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all $\mathbf{u} \in V$ and all $c,\ d \in \mathbb{R}$.
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

Examples

1. $\mathbb{R}^n$ with the usual vector addition and scalar multiplication.
2. $C[0,1]$ is the vector space of all continuous real-valued functions defined on the interval $[0,1] \subset \mathbb{R}$. The vector sum of two functions f and g is the function $f + g$ defined by

   $$(f + g)(x) = f(x) + g(x)$$

   for all $x \in [0,1]$. (Note that the sum of two continuous functions is continuous.) The scalar multiple cf of $c \in \mathbb{R}$ with $f \in C[0,1]$ is the function defined by

   $$(cf)(x) = c(f(x)).$$

   Again you should note that cf defined in this manner is continuous if f is continuous.

   Property 3 requires the existence of a “zero vector”. For $C[0,1]$, the zero vector is the function that is constantly zero for all $x \in C[0,1]$.

Just as a vector space is a generalization of $\mathbb{R}^n$, there is a generalization of the concept of the dot product. A generalized dot product is often called an inner product.

Definition of an Inner Product
Let V be a vector space. An **inner product** on V is a scalar-valued function defined on $V \times V$ (ordered pairs of elements of V). The inner product of two vectors $\mathbf{u},\ \mathbf{v} \in V$ is often denoted $< \mathbf{u}, \mathbf{v} >$, and it must satisfy the following four properties:

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$ for all $\mathbf{u},\ \mathbf{v} \in V$.
2. $< \mathbf{u} + \mathbf{v}, \mathbf{w} >=< \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$ for all $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$.
3. $< c\mathbf{u}, \mathbf{v} >= c < \mathbf{u}, \mathbf{v} >=< \mathbf{u}, c\mathbf{v} >$ for all $\mathbf{u},\ \mathbf{v} \in V$ and all $c \in \mathbb{R}$.
4. $< \mathbf{u}, \mathbf{u} > \geq 0$ for all $\mathbf{u} \in V$, and $< \mathbf{u}, \mathbf{u} >= 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Examples

1. The usual dot product on $\mathbb{R}^n$.
2. Given $f,\ g \in C[0, 1]$, define

$$< f, g >= \int_0^1 f(x)g(x)\, dx.$$

   A major portion of Exercise 15 involves verifying that $< f, g >$ defines an inner product on $C[0, 1]$. The most important item is the second half of the fourth property.

Every vector space with an inner product has an associated norm (a length function):

Definition of a Norm from an Inner Product
Given an inner product $< \ \ , \ \ >$ for the vector space V, we define the **norm** $||\mathbf{v}||$ of a vector $\mathbf{v} \in V$ by

$$||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}.$$

Properties of a Norm
From the properties of an inner product, we observe that the associated norm satisfies four properties:

1. $||\mathbf{v}|| \geq 0$ for all $\mathbf{v} \in V$.
2. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
3. $||c\mathbf{v}|| = |c|||\mathbf{v}||$ for all $c \in \mathbb{R}$ and all $\mathbf{v} \in V$.

4. $||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ for all $\mathbf{u},\ \mathbf{v} \in V$ (triangle inequality).

The fact that the norm satisfies the (extremely important) triangle inequality follows from the Cauchy-Schwarz inequality.

Cauchy-Schwarz Inequality Let $< \ , \ >$ be an inner product on the vector space V. Then

$$| < \mathbf{u}, \mathbf{v} > | \leq ||\mathbf{u}||\, ||\mathbf{v}||$$

for all $\mathbf{u},\ \mathbf{v} \in V$.

We can prove the Cauchy-Schwarz inequality just by mimicking the proof for $\mathbb{R}^n$ on page 66.

Proof. If either $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, then there is nothing to prove.
Let $a,\ b \in \mathbb{R}$. We have

$$0 \leq ||a\mathbf{u} + b\mathbf{v}||^2 = < a\mathbf{u} + b\mathbf{v}, a\mathbf{u} + b\mathbf{v} >$$

$$= a^2||\mathbf{u}||^2 + 2ab < \mathbf{u}, \mathbf{v} > + b^2||\mathbf{v}||^2.$$

Now use $a = ||\mathbf{v}||^2$ and $b = - < \mathbf{u}, \mathbf{v} >$. With these values of a and b, we get

$$0 \leq ||\mathbf{u}||^2||\mathbf{v}||^4 - 2 < \mathbf{u}, \mathbf{v} >^2 ||\mathbf{v}||^2 + < \mathbf{u}, \mathbf{v} >^2 ||\mathbf{v}||^2$$

$$= ||\mathbf{u}||^2||\mathbf{v}||^4 - < \mathbf{u}, \mathbf{v} >^2 ||\mathbf{v}||^2.$$

Moving $< \mathbf{u}, \mathbf{v} >^2 ||\mathbf{v}||^2$ to the left-hand side of the inequality, we have

$$< \mathbf{u}, \mathbf{v} >^2 ||\mathbf{v}||^2 \leq ||\mathbf{u}||^2||\mathbf{v}||^4.$$

Dividing both sides by $||\mathbf{v}||^2$ yields

$$< \mathbf{u}, \mathbf{v} >^2 \leq ||\mathbf{u}||^2||\mathbf{v}||^2,$$

and we obtain the Cauchy-Schwarz inequality by taking the square root of both sides. ■

Now that we have the Cauchy-Schwarz inequality, we can derive the triangle inequality as follows:
Given $\mathbf{u},\ \mathbf{v} \in V$, we have

$$||\mathbf{u} + \mathbf{v}||^2 = < \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} >^2$$

$$= ||\mathbf{u}||^2 + 2 < \mathbf{u}, \mathbf{v} > + ||\mathbf{v}||^2.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$||\mathbf{u} + \mathbf{v}||^2 \leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\, ||\mathbf{v}|| + ||\mathbf{v}||^2$$

$$= (||\mathbf{u}|| + ||\mathbf{v}||)^2.$$

We obtain the triangle inequality by taking the square root of both sides.

Given a norm on V, we obtain a distance function (a metric) by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||.$$