Another Definition of the Determinant

In order to give a definition of the determinant without using recursion, we need to assemble a few facts about permutations of finite sets of natural numbers.

Definition. A *permutation* of the set $\{1, 2, 3, \ldots, n\}$ is a one-to-one function

$$\sigma : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}.$$

In other words, it is a rearrangement of the set $\{1, 2, \ldots, n\}$.

Example. The following table represents a permutation of the first six natural numbers.

i	1	2	3	4	5	6
$\sigma(i)$	2	4	5	6	3	1

Definition. The *transposition* (i, j) is the permutation that interchanges i and j and leaves the other numbers fixed.

Theorem. *Every permutation is a combination of transpositions.*

Example. Consider the permutation σ above. We can obtain σ as the following sequence of transpositions. First, perform the transposition $(1, 2)$, then $(1, 4)$, then $(1, 6)$, and finally $(3, 5)$. The resulting permutation is σ.

Theorem. *If one representation of a permutation by transpositions involves an even number of transpositions, then all representations involve an even number of transpositions. Therefore, an analogous statement applies to the combination of an odd number of transpositions.*

Definition. The *sign* of a permutation σ, denoted $\operatorname{sign}(\sigma)$, is $+1$ if σ is a combination of an even number of transpositions and is -1 if σ is a combination of an odd number of transpositions.

Example. The sign of the permutation σ in the above example is $+1$ because we can represent σ as the product of four transpositions.

Given all of this formalism, we can now write down a formula for the determinant in terms of permutations.

Definition. Let $\mathbf{A}$ be an $n \times n$ matrix. Then

$$\det \mathbf{A} = \sum \operatorname{sign}(\sigma) \left(a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} \right)$$

where there is one term in the sum for each permutation σ of $\{1, 2, \ldots, n\}$.

Remark. Since there are $n!$ different permutations of $\{1, 2, \ldots, n\}$, the above definition consists of the sum of $n!$ numbers, each of which is the product of n entries of the matrix $\mathbf{A}$.