1. (12 points) Row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 & 3 & 1 \\ 3 & 9 & -1 & 7 & 3 \\ -2 & -6 & 4 & -8 & -1 \end{bmatrix}$$

to **reduced row echelon form** (RREF). Do only one row operation at a time and specify that operation when you perform it. Indicate when you first arrive at a matrix in **echelon form** (REF). What are the pivot positions of $\mathbf{A}$?

2. (16 points) Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + hx_2 + x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= k, \end{aligned}$$

where h and k are real numbers. Determine all values of h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Answer each part separately.

3. (12 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Determine the standard matrix representation for T.

(b) Calculate $T(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

4. (14 points)

(a) What is a nontrivial dependence relation among a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ \ldots, \ \mathbf{v}_k\}$?

(b) We know that the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix} \right\}$$

is linearly dependent. What are all of the possible dependence relations among this set of vectors? (Your final answer should be expressed as efficiently as possible. In other words, the relations should be expressed in terms of as few parameters as possible.)

5. (16 points) Consider the following eight 2×2 matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

Each matrix defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Exactly one defines a dilation. Exactly one defines a projection. Exactly one defines a rotation, and exactly one defines a shear. Match the matrix with its geometric description, and provide a brief justification for your choice. **You will not receive any credit unless you justify your selection.**

(a) The matrix for the dilation is ____. My reason for choosing this answer is:

(b) The matrix for the projection is ____. My reason for choosing this answer is:

(c) The matrix for the rotation is ____. My reason for choosing this answer is:

(d) The matrix for the shear is ____. My reason for choosing this answer is:

6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

   (a) The equation $\mathbf{Ax} = \mathbf{b}$ is consistent if the augmented matrix $[\mathbf{A} \quad \mathbf{b}]$ has a pivot position in every row.

   (b) The columns of any 4×3 matrix are linearly dependent.

Question 6 (continued):

(c) Let T be a linear transformation. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

(d) Let $\mathbf{A}$ be an $m \times n$ matrix. The range of the linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is the set of all linear combinations of the columns of $\mathbf{A}$.

Question 6 (continued):

(e) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(f) If $\mathbf{A}$ is an invertible $n \times n$ matrix, then the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^n$.