1. (12 points) Row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 & 3 & 1 \\ 3 & 9 & -1 & 7 & 3 \\ -2 & -6 & 4 & -8 & -1 \end{bmatrix}$$ pivot positions

to **reduced row echelon form** (RREF). Do only one row operation at a time and specify that operation when you perform it. Indicate when you first arrive at a matrix in **echelon form** (REF). What are the pivot positions of **A**?

$$A \underset{R_2 \to R_2 - 3R_1}{\sim} \begin{bmatrix} 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ -2 & -6 & 4 & -8 & -1 \end{bmatrix} \underset{R_3 \to R_3 + 2R_1}{\sim}$$

$$\begin{bmatrix} 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 & 1 \end{bmatrix} \underset{R_3 \to R_3 - R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

REF

$$\underset{R_1 \to R_1 - R_3}{\sim} \begin{bmatrix} 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to \frac{1}{2} R_2}{\sim}$$

$$\begin{bmatrix} 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_1 \to R_1 + R_2}{\sim}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF

2. (16 points) Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + hx_2 + x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= k, \end{aligned}$$

where h and k are real numbers. Determine all values of h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Answer each part separately.

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & h & 1 & 4 \\ 1 & 2 & 3 & k \end{array}\right]$

Row reduce: $R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & h-6 & -2 & -2 \\ 0 & 0 & 2 & k-2 \end{array}\right]$$

If $h \neq 6$, the system is consistent and has a unique solution.

If $h = 6$, we apply the row op $R_3 \to R_3 + R_2$ to get $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & k-4 \end{array}\right]$.

If $k \neq 4$, the system is inconsistent

If $k = 4$, the system is consistent and has one free variable.

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#2 (cnt).

(a) no solution if $h=6$ and $k \neq 4$

(b) unique solution if $h \neq 6$

(c) many solutions if $h=6$ and $k=4$.

3. (12 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$\overset{e_1}{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

(a) Determine the standard matrix representation for T.

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow T(e_2) = \frac{1}{2} T\begin{bmatrix} 2 \\ 2 \end{bmatrix} - T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

matrix: $\begin{bmatrix} 7 & -6 \\ -3 & 3 \\ 1 & 1 \end{bmatrix}$

(b) Calculate $T(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

$$T(v) = \begin{bmatrix} 7 & -6 \\ -3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -23 \\ 12 \\ 6 \end{bmatrix}$$

4. (14 points)

(a) What is a nontrivial dependence relation among a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$?

A nontrivial dependence relation is $r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0$ where $r_i \in \mathbb{R}$ and at least one $r_i \neq 0$.

(b) We know that the set of vectors

$$\left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}} \right\}$$

is linearly dependent. What are all of the possible dependence relations among this set of vectors? (Your final answer should be expressed as efficiently as possible. In other words, the relations should be expressed in terms of as few parameters as possible.)

$$r_1 v_1 + r_2 v_2 + r_3 v_3 + r_4 v_4 = \underbrace{\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 3 & 9 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{Ar}$$

Want $Ar = 0$. We row reduce A.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

We get $r_3 = 2r_4$, $r_2 = r_4 - 2r_3$

and $r_1 = -2r_2 - 3r_3 - 3r_4$.

Express in terms of the free variable r_4:

$r_3 = 2r_4$ $\quad$ $r_2 = r_4 - 2(2r_4) = -3r_4$ $\quad$ $r_1 = -2r_2 - 3r_3 - 3r_4 = 6r_4 - 6r_4 - 3r_4 = -3r_4$

5. (16 points) Consider the following eight 2×2 matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

Each matrix defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Exactly one defines a dilation. Exactly one defines a projection. Exactly one defines a rotation, and exactly one defines a shear. Match the matrix with its geometric description, and provide a brief justification for your choice. **You will not receive any credit unless you justify your selection.**

(a) The matrix for the dilation is C. My reason for choosing this answer is:

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix}$$

expands by a factor of 7 in every direction.

(b) The matrix for the projection is A. My reason for choosing this answer is:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

projection onto the x_2 - axis.

(c) The matrix for the rotation is D. My reason for choosing this answer is:

Rotations have the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

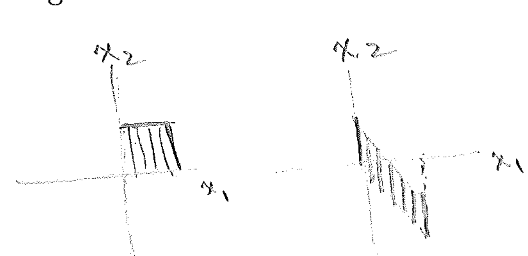
Here $\cos\theta = -0.6$

$\sin\theta = -0.8$

Note that $(-.6)^2 + (-.8)^2 = 1$.

(d) The matrix for the shear is F. My reason for choosing this answer is:

$$F \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - 3x_1 \end{bmatrix}$$



6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

   (a) The equation $\mathbf{Ax} = \mathbf{b}$ is consistent if the augmented matrix $[\mathbf{A} \quad \mathbf{b}]$ has a pivot position in every row.

   False. If any pivot position is in the last column of the augmented matrix, then the system is inconsistent.

   (b) The columns of any 4×3 matrix are linearly dependent.

   False. The columns of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   are linearly independent.

Question 6 (continued):

(c) Let T be a linear transformation. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

True. If $\{v_1, v_2, v_3\}$ is linearly dependent, there is a nontrivial dependence relation $r_1 v_1 + r_2 v_2 + r_3 v_3 = 0$. Apply T to this relation and get

$$T(r_1 v_1 + r_2 v_2 + r_3 v_3) = T(0) = 0$$

$$r_1 T(v_1) + r_2 T(v_2) + r_3 T(v_3) = 0$$

This is a nontrivial dep. relation for $\{T(v_1), T(v_2), T(v_3)\}$.

(d) Let $\mathbf{A}$ be an $m \times n$ matrix. The range of the linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is the set of all linear combinations of the columns of $\mathbf{A}$.

True. The range is the set of all vectors of the form Ax for all $x \in \mathbb{R}^n$. If $A = [A_1 | A_2 | \cdots | A_n]$, then Ax is $x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$. This is precisely the set of all linear combinations of the columns of A (the span of $\{A_1, A_2, \ldots, A_n\}$).

Question 6 (continued):

(e) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

False. $(A+B)(A-B) = A^2 - AB + BA - B^2$.
This equals $A^2 - B^2$ only if $-AB + BA = 0$.
In other words, AB would have to be equal to BA. This is true for some pairs of matrices but not for all pairs.

(f) If $\mathbf{A}$ is an invertible $n \times n$ matrix, then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^n$.

True. If A^{-1} exists, then apply it to both sides of the equation: $A^{-1}Ax = A^{-1}b$

$$\Rightarrow x = A^{-1}b.$$