1. (12 points) **Use row operations** to calculate the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 3 & -1 & 4 & 6 \\ 2 & -2 & 2 & 4 \\ -2 & 4 & -1 & -1 \end{bmatrix}.$$

$$A \sim \begin{bmatrix} 2 & -2 & 2 & 4 \\ 3 & -1 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ -2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -1 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ -2 & 4 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = B$$

All row operations were row replacements except for one flip and one scale. Since $\det B = 6$, we have $\det A = (-1)(2)(6) = -12$.

2. (16 points) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 \\ 1 & 1 & 3 & 2 & -1 \end{bmatrix}.$$

Calculate bases for col **A** and nul **A**.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns are 1, 3, 5 $\Rightarrow$

basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \right\}$

To find a basis for nul A, we solve $Ax = 0$. the variables $x_2$ and $x_4$ are free. We get $x_5 = 0$, $x_3 = -x_4$, and

$x_1 = -x_2 - x_3 = -x_2 + x_4$.

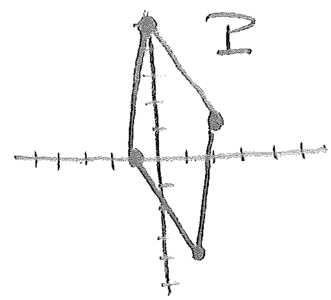
$$\text{nul}A = \begin{bmatrix} -x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

basis of nul A is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

3. (10 points) Let $P$ be the parallelogram in $\mathbb{R}^2$ with vertices $(-1,0)$, $(0,5)$, $(1,-4)$, $(2,1)$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(\mathbf{x}) = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}.$$

Calculate the area of $T(P)$.




Translate P so that one vertex is at the origin: We add $(1,0)$. Then the other vertices are $(1,5)$, $(2,-4)$, $(3,1)$. The area of P is

$$\left| \det \begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix} \right| = |-14| = 14.$$

The area of $T(P)$ is $|\det T|(\text{area } P)$.

$$\text{area of } T(P) = |5-2|(14)$$
$$= (3)(14) = 42.$$

4. (16 points) **Note that part b of this problem is on the next page.** Let

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 3 & 0 & 6 \end{bmatrix}.$$

(a) Compute $\mathbf{A}^{-1}$. You may use your calculator to double check your answer, but you will not get any credit unless you show enough work so that I can be sure that you can do this problem without your calculator.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 3 & 0 & 6 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

4. (continued)

(b) Write $\mathbf{A}^{-1}$ as a product of elementary matrices. You do **NOT** need to multiply the elementary matrices together when you write $\mathbf{A}^{-1}$ as a product.

Each row operation is one elementary matrix:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the previous page, we have

$$E_5 E_4 E_3 E_2 E_1 A = I$$

$$\Rightarrow A^{-1} = E_5 E_4 E_3 E_2 E_1$$

5. (16 points) **Note that part b of this problem is on the next page.** The trace of a matrix is the sum of its entries along the diagonal. For example, the trace of the $2 \times 2$ matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $a + d$. Consider the subset $S$ of the vector space $M_{2\times2}$ of all $2 \times 2$ matrices that consists of all matrices whose trace is zero.

(a) Show that $S$ is a vector subspace of $M_{2\times2}$.

(1) contains the zero vector (not necessary):

the trace of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is $0$. ✓

(2) closure under vector addition:

$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$a_1 + d_1 = 0$ $\quad a_2 + d_2 = 0$

$\Rightarrow A_1 + A_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

Since $d_1 + d_2 = -(a_1 + a_2)$, the trace of $A_1 + A_2$ is zero. ✓

(3) closure under scalar multiplication:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $d = -a$.

Let $r \in \mathbb{R}$. Then $rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$.

trace $(rA) = ra + rd = r(a+d) = 0$ ✓

Problem 5 (continued):

(b) Determine a basis for $S$. Justify that your answer is a basis.

Need a linearly independent, spanning set.

Use the matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

and $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(1) These three matrices are linearly independent:

Consider the dependence relation

$r_1 A_1 + r_2 A_2 + r_3 A_3 = 0 \Rightarrow$

$\begin{bmatrix} r_1 & r_2 \\ r_3 & -r_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow r_1 = r_2 = r_3 = 0$

The relation is trivial.

(2) These three matrices span $S$:

Let $A \in S$. Then $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

$A = aA_1 + bA_2 + cA_3$.

$\Rightarrow S = \text{span}\{A_1, A_2, A_3\}$.

6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

(a) The plane $x_1 + x_2 - 2x_3 = 1$ is a subspace of $\mathbb{R}^3$.

False. The zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not lie on the plane.

$= A$

(b) If the columns of an $n \times n$ matrix are linearly independent, then so are the rows of the matrix.

True. If the columns are linearly independent, then $A$ is invertible. By the IMT, $A^T$ is invertible.

$\Rightarrow$ columns of $A^T$ are linearly independent.

$\Rightarrow$ rows of $A$ are linearly independent.

Question 6 (continued):

(c) If $\det(2\mathbf{A}) = 0$ for an $n \times n$ matrix $\mathbf{A}$, then $\mathbf{A}$ is not invertible.

True. $\det(2A) = 2^n(\det A)$.

If $\det(2A) = 0$, then $\det A = 0$,

and $A$ is not invertible.

(d) If $H$ and $K$ are subspaces of a vector space $V$, then their union $H \cup K$ is a subspace of $V$.

False. Consider the two subspaces of $\mathbb{R}^2$ defined by $x_2 = x_1$ and $x_2 = -x_1$. Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $v_1 + v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in the union of the two subspaces.

Not closed under vector addition.

Question 6 (continued):

(e) The transpose of an elementary matrix is elementary. True.

There are three types of elementary matrices corresponding to the three types of row operations.

1. scale a row ($E_s$): $E_s$ is a diagonal matrix and $E_s^T = E_s$.
2. interchange two rows $R_i \leftrightarrow R_j$: the matrix $E_f$ is a diagonal matrix except on rows $i$ and $j$ where $(E_f)_{i,j} = 1$ and $(E_f)_{j,i} = 1$. In this case, $(E_f)^T = E_f$.
3. row replacement ($E_r$): $R_i \to R_i + \alpha R_j$, $\alpha \neq 0$

Then $E_r$ is the identity matrix with exactly one more nonzero entry, $(E_r)_{i,j} = \alpha$. Then $(E_r)^T$ is the identity matrix with exactly one more nonzero entry $(E_r^T)_{j,i} = \alpha$. So $(E_r)^T$ corresponds to the row operation $R_j \to R_j + \alpha R_i$.