1. (14 points) Diagonalize the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -8 \\ 1 & -1 \end{bmatrix}.$$

In other words, write $\mathbf{A}$ as $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix.

2. (14 points) In order to receive any credit, you must provide a brief justification for each answer.

   (a) Suppose that a 6×9 matrix $\mathbf{A}$ has four pivot columns. What are the dimensions of Nul $\mathbf{A}$, Col $\mathbf{A}$, and Row $\mathbf{A}$?

   (b) What is the smallest possible dimension of the null space of a 6×9 matrix?

   (c) Suppose that a 9×6 matrix $\mathbf{A}$ has four pivot columns. What are the dimensions of Nul $\mathbf{A}$, Col $\mathbf{A}$, and Row $\mathbf{A}$?

   (d) What is the smallest possible dimension of the null space of a 9×6 matrix?

3. (16 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} -7 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find the point closest to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Calculate the distance of $\mathbf{y}$ to W.

4. (16 points) Calculate an orthogonal basis for the row space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 3 & -5 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}.$$

5. (10 points) What can be said about a matrix $\mathbf{A}$ that is similar to the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}?$$

Provide a brief explanation of each of your assertions.

6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

   (a) If there exists a linearly-dependent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V, then $\dim V \leq p - 1$.

   (b) Each eigenspace of a square matrix $\mathbf{A}$ is the null space of some matrix.

Question 6 (continued):

(c) Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$. If $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is in $W^\perp$.

(d) For a square matrix $\mathbf{A}$, an eigenvector $\mathbf{v}$ of $\mathbf{A}$ is also an eigenvector of $\mathbf{A}^2$.

Question 6 (continued):

(e) For a square matrix $\mathbf{A}$, the vectors in Col $\mathbf{A}$ and the vectors in Nul $\mathbf{A}$ are orthogonal.

(f) For a square matrix $\mathbf{A}$, if $\mathbf{A}^2$ is diagonalizable, then $\mathbf{A}$ is diagonalizable.