1. (14 points) Diagonalize the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -8 \\ 1 & -1 \end{bmatrix}.$$

In other words, write $\mathbf{A}$ as $\mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix.

Char. poly $\det \begin{bmatrix} 5-\lambda & -8 \\ 1 & -1-\lambda \end{bmatrix} = (\lambda+1)(\lambda-5)+8$

$= \lambda^2 - 4\lambda + 3$

$= (\lambda-3)(\lambda-1)$

Eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 1$

$\lambda=3$ espace $= \mathrm{nul} \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$

$\lambda = 1$ espace $= \mathrm{nul} \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

One $P = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad \det P = 4-2 = 2$

$\Rightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$

2. (14 points) In order to receive any credit, you must provide a brief justification for each answer.

   (a) Suppose that a 6×9 matrix $\mathbf{A}$ has four pivot columns. What are the dimensions of Nul $\mathbf{A}$, Col $\mathbf{A}$, and Row $\mathbf{A}$?

   four pivot cols $\Rightarrow$ rank $A = 4$
   $\Rightarrow$ dim col$A$ = dim row$A = 4$

   rank + dim nul$A = 9 \Rightarrow$ dim nul$A = 5$

   (b) What is the smallest possible dimension of the null space of a 6×9 matrix?

   rank + dim nul$A = 9$
   largest rank $= 6 \Rightarrow$ smallest
   dim nul$A = 3$.

   (c) Suppose that a 9×6 matrix $\mathbf{A}$ has four pivot columns. What are the dimensions of Nul $\mathbf{A}$, Col $\mathbf{A}$, and Row $\mathbf{A}$?

   four pivot cols $\Rightarrow$ rank $= 4$
   $\Rightarrow$ dim col$A$ = dim row$A$
   $= 4$

   rank + dim nul$A = 6 \Rightarrow$ dim nul$A = 2$

   (d) What is the smallest possible dimension of the null space of a 9×6 matrix?

   rank + dim nul$A = 6$
   largest rank $= 6 \Rightarrow$
   smallest dim nul$A = 0$.

3. (16 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} -7 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find the point closest to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$V_1 \cdot V_2 = 2+2+9+2 = 15$

Need orthogonal basis so we calculate

$V_3 = V_2 - \left(\frac{V_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1 \qquad V_1 \cdot V_1 = 15$

$V_3 = V_2 - \left(\frac{15}{15}\right) V_1 = V_2 - V_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

point closest = $\text{proj}_W Y$

$= \left(\frac{Y \cdot V_1}{V_1 \cdot V_1}\right) V_1 + \left(\frac{Y \cdot V_3}{V_3 \cdot V_3}\right) V_3 = \left(\frac{10}{15}\right) V_1 + \left(\frac{7}{3}\right) V_3$

$= \frac{2}{3} V_1 + \frac{7}{3} V_3$

$= \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ -2 \\ \frac{4}{3} \end{bmatrix} + \begin{bmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ 0 \\ -\frac{7}{3} \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 3 \\ -2 \\ -1 \end{bmatrix}$

(b) Calculate the distance of $\mathbf{y}$ to W.

distance $= \| Y - \text{proj}_W Y \|$

$= \left\| \begin{bmatrix} -4 \\ 0 \\ 4 \\ 4 \end{bmatrix} \right\| = 4 \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\|$

$= 4\sqrt{3}$

4. (16 points) Calculate an orthogonal basis for the row space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 3 & -5 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}.$$

Start with a basis of Row A - row reduce

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 4 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

First two rows form a basis of row A.

Now produce an orthogonal basis

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix} \quad v_1 \cdot v_2 = -9$$

Calculate $v_3 = v_2 - \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$$= v_2 - \frac{-9}{7} v_1$$

$$= \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix} + \frac{9}{7}\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ \frac{5}{7} \\ -\frac{3}{7} \\ -\frac{2}{7} \end{bmatrix}$$

Could also use

$$v_3 = \begin{bmatrix} 9 \\ 5 \\ -3 \\ -2 \end{bmatrix}$$

orthogonal basis $= \{v_1, v_3\}$

5. (10 points) What can be said about a matrix $\mathbf{A}$ that is similar to the diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}?$$

Provide a brief explanation of each of your assertions.

Similar matrices have the same char poly

So the char poly of $A = (3-\lambda)^3(6-\lambda)^2(7-\lambda)$.

The eigenvalues of A are $\lambda = 3$ (alg mult = 3), $\lambda = 6$ (alg mult = 2), and $\lambda = 7$ (alg mult = 1).

The dim of espaces remain the same for similar matrices.

$\lambda = 3$ espace of D = Span$\{e_1, e_3, e_4\}$

$\lambda = 6$ espace of D = Span$\{e_2, e_5\}$

$\lambda = 7$ espace of D = Span$\{e_6\}$.

$\Rightarrow$ dim of $\lambda = 3$ espace of A is 3

dim of $\lambda = 6$ espace of A is 2

dim of $\lambda = 7$ espace of A is 1

Also, $\det A = \det D = (3^3)(6^2)(7) = 6804$.

Hence, A is invertible as well as diagonalizable.

6. (30 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are four more parts to this question on the next two pages.)

   (a) If there exists a linearly-dependent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V, then $\dim V \leq p - 1$.

   False. For example, $\mathbb{R}^3$ contains the linearly dependent set $\{e_1, 2e_1\}$ with two vectors. However, $\dim \mathbb{R}^3 = 3$.

   (b) Each eigenspace of a square matrix $\mathbf{A}$ is the null space of some matrix.

   True. The λ eigenspace is the set of all vectors x such that $Ax = \lambda x$. If $Ax = \lambda x$, then $(A - \lambda I)x = 0 \Rightarrow$ the λ-espace is nul B where B is the matrix $(A - \lambda I)$.

Question 6 (continued):

(c) Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$. If $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is in $W^\perp$.

True. Any vector w in W is a linear combination of w_1 and w_2. Compute

$$v \cdot w = v \cdot (c_1 w_1 + c_2 w_2)$$
$$= c_1(v \cdot w_1) + c_2(v \cdot w_2) = 0$$

Therefore, $v \cdot w = 0$ for all w in W.

$\Rightarrow v$ is in $W^\perp$.

(d) For a square matrix $\mathbf{A}$, an eigenvector $\mathbf{v}$ of $\mathbf{A}$ is also an eigenvector of $\mathbf{A}^2$.

True. Suppose that $Av = \lambda v$ for some scalar λ and nonzero v.

Then
$$A^2(v) = A(Av) = A(\lambda v)$$
$$= \lambda(Av)$$
$$= \lambda(\lambda v)$$
$$= \lambda^2 v.$$

So v is an eigenvector of A^2 corresponding to the eigenvalue λ^2 of A^2.

Question 6 (continued):

(e) For a square matrix **A**, the vectors in Col **A** and the vectors in Nul **A** are orthogonal.

False. $\text{Nul} A = (\text{Row} A)^\perp$.

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a vector in Nul A. However, $\text{Col} A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ but $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1$. Hence, Nul A and Col A are not orthogonal.

(f) For a square matrix **A**, if $\mathbf{A}^2$ is diagonalizable, then **A** is diagonalizable.

False. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that A corresponds to rotation by 90°. Hence, A does not have any eigenvectors. However, A^2 is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and this matrix is diagonal.