

## Determinants

We start with a recursive definition of the determinant.

**Definition.** The determinant of a  $1 \times 1$  matrix  $[a_{11}]$  is  $a_{11}$ .

Now we define the determinant of an  $n \times n$  matrix in terms of determinants of  $(n-1) \times (n-1)$  matrices.

**Definition.** Given an  $n \times n$  matrix  $\mathbf{A}$ , the  $ij$ th minor  $\mathbf{A}_{ij}$  of  $\mathbf{A}$  is the  $(n-1) \times (n-1)$  matrix obtained from  $\mathbf{A}$  by eliminating the  $i$ th row and  $j$ th column. The  $ij$ th cofactor of  $\mathbf{A}$  is

$$C_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}.$$

**Example.** Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -1 \\ 4 & 0 & 2 \end{bmatrix}.$$

**Definition/Theorem.** If  $\mathbf{A}$  is an  $n \times n$  matrix, the determinant of  $\mathbf{A}$  can be computed using cofactor expansion along the  $i$ th row by

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

or by cofactor expansion along the  $j$ th column by

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$

for any row or any column.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Is there a way to define  $\det \mathbf{A}$  without recursion?

How do we go about computing  $\det \mathbf{A}$ ?

One type of matrix is perfectly suited for cofactor expansion.

**Theorem.** If  $\mathbf{A}$  is a triangular matrix, then  $\det \mathbf{A}$  is the product of its entries along the main diagonal.

In order to gain some insight into how we will compute determinants in general, let's calculate the determinants of all elementary  $3 \times 3$  matrices.