Geometry of the Determinant

Consider a parallelogram P in $\mathbb{R}^2$ formed by two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$.

Example. Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let's calculate the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ in an unusual way.

Summary: 2×2 Matrices and Area in $\mathbb{R}^2$

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is

$$\pm \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

Volume in $\mathbb{R}^3$

The same basic geometric argument that works in $\mathbb{R}^2$ also works in $\mathbb{R}^3$. Given three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$, consider the parallelopiped that they determine.

How do column operations affect the volume of the corresponding parallelopipeds?

Summary: 3×3 Matrices and Volume in $\mathbb{R}^3$

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelopiped formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is

$$\pm \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Example. Consider the parallelopiped generated by

$$\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix},$$

Determinants and linear transformations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a matrix $\mathbf{A}$ such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

What is the significance of $\det \mathbf{A}$ in this situation?

Consider the case where $n = 2$ and start with a parallelogram P determined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

Note. There is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk. For more details, see pp. 208–209 of our text.