

Instant replay regarding the geometry of determinants

Summary: 2×2 Matrices and Area in \mathbb{R}^2

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then the area of the parallelogram formed by \mathbf{u} and \mathbf{v} is

$$\left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|.$$

Summary: 3×3 Matrices and Volume in \mathbb{R}^3

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelepiped formed by \mathbf{a} , \mathbf{b} , and \mathbf{c} is

$$\left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|.$$

Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then there exists a matrix \mathbf{A} such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

For $n = 2$, consider a parallelogram P determined by two vectors \mathbf{u} and \mathbf{v} . Then $T(P)$ is also a parallelogram. In this case,

$$\text{area of } T(P) = (\det A)(\text{area of } P).$$

For $n = 3$, the same conclusion holds if the concept of area is replaced by that of volume.

Also, there is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk. For more details, see pp. 208–209 of our text.

What is a vector space?

Definition. A vector space V is a set of objects that are called vectors along with two operations—vector addition and scalar multiplication. The vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is always defined for any pair of vectors \mathbf{v}_1 and \mathbf{v}_2 in V , and given any scalar r in \mathbb{R} and any vector \mathbf{v} in V , the scalar multiple $r\mathbf{v}$ is a vector in V . Moreover, these two operations must satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each \mathbf{u} , there is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Example 1. The vector space \mathbb{R}^n . See p. 32 of our text a discussion of the properties listed above.

Example 2. The vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Example 3. The vector space $M_{m \times n}$ of all $m \times n$ matrices. (We assume that the entries of the matrices are real numbers, but a different vector space is obtained if one allows the entries to be complex numbers.)

Example 4. The vector space of all polynomial functions $p : \mathbb{R} \rightarrow \mathbb{R}$.

Example 5. The vector space of all discrete-time signals.

Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S ,
2. (closure under vector addition) for each \mathbf{v}_1 and \mathbf{v}_2 in S , the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S , and
3. (closure under scalar multiplication) for each r in \mathbb{R} and each \mathbf{v} in S , the scalar multiple $r\mathbf{v}$ is in S .

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space \mathbb{R}^2 .

Example. Consider the line $x_2 = x_1 + 1$ in the vector space \mathbb{R}^2 .