Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$.

Example. Consider the line $x_2 = x_1 + 1$ in the vector space $\mathbb{R}^2$.

Example. Let $\mathbb{P}$ represent the vector space of all polynomial functions as discussed last class. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is S a subspace of $\mathbb{P}$?

Theorem. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Example. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V?

1. The set of all constant functions.

2. The set of all functions f such that $f(2) = 1$.

3. The set of all functions f such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.

Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $\mathbf{A}$.

The null space of A. The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

The null space of $\mathbf{A}$ is denoted by Nul $\mathbf{A}$.

Theorem. The null space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

Application. Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 2 & -4 & 0 & 8 & 1 \end{bmatrix}.$$

Express the null space of $\mathbf{A}$ as the span of as few vectors as possible.