Bases for vector spaces and subspaces

Given a vector space or subspace V, we often find it convenient to express it as the span of a few vectors. A basis for V is a spanning set that contains as few vectors as possible.

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a *basis* for V if

1. it is linearly independent, and
2. it spans V.

Example. The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$.

Example. The two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a basis of $\mathbb{R}^2$.

Example. The set $\{x^3, x^2, x, 1\}$ is a basis of $\mathbb{P}_3$.

Example. The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of $\mathbb{P}_3$.

We need ways of determining bases of vector spaces and their subspaces. The "casting-out procedure" produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace S spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, we can obtain a basis B for S by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
   (a) $B_i = B_{i-1}$ if $\mathbf{v}_i$ is in $\text{Span}B_{i-1}$, or
   (b) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if $\mathbf{v}_i$ is not in $\text{Span}B_{i-1}$.

Then the final result B_k is a basis for S.

Example. Let's apply the casting-out procedure to the set

$$\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$$

of polynomials in $\mathbb{P}_3$.

Theorem. (similar to The Spanning Set Theorem, Lay, p. 239) Let $S = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then the final result B_k of the casting-out procedure applied to $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for S.

To prove this theorem, we must show that the casting-out procedure produces a linearly independent set that still spans S.

Linear independence: Let B_i be the first step in the procedure for which B_i is linearly dependent.

Spanning: We must show that $\text{Span} B_k = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Bases for Nul $\mathbf{A}$ and Col $\mathbf{A}$

Last class we did an example that showed how we can produce a basis for Nul $\mathbf{A}$.

Example. Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example. Find a basis for the column space of

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$