

The Invertible Matrix Theorem

Theorem. Let \mathbf{A} be an $n \times n$ matrix. Then the following twelve statements are equivalent:

- (a) \mathbf{A} is an invertible matrix.
- (b) \mathbf{A} is row equivalent to the identity matrix.
- (c) \mathbf{A} has n pivot positions
- (d) The equation $\mathbf{Ax} = \mathbf{0}$ has no nontrivial solutions.
- (e) The columns of \mathbf{A} are linearly independent.
- (f) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ is one-to-one.
- (g) The equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.
- (h) The columns of \mathbf{A} span \mathbb{R}^n .
- (i) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- (j) There is an $n \times n$ matrix \mathbf{C} such that $\mathbf{CA} = \mathbf{I}$.
- (k) There is an $n \times n$ matrix \mathbf{D} such that $\mathbf{AD} = \mathbf{I}$.
- (l) \mathbf{A}^T is an invertible matrix.

Computer graphics

Homogeneous coordinates are useful when we want to do computer graphics with matrices.

Definition. A point (x, y) in \mathbb{R}^2 can be represented by the point $(x, y, 1)$ in \mathbb{R}^3 . The coordinates $(x, y, 1)$ are called the homogeneous coordinates of the point (x, y) .

Homogeneous coordinates are useful because translation in \mathbb{R}^2 can be represented by a linear transformation in \mathbb{R}^3 .

Fact 1. A translation by (h, k) in \mathbb{R}^2 can be obtained by matrix multiplication of homogeneous coordinates. That is,

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}.$$

Fact 2. Any linear transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be represented as a transformation of homogeneous coordinates by matrix multiplication. In particular, if the transformation is represented by the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the corresponding matrix for homogeneous coordinates is

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example. What is the matrix that represents rotation by 45° in terms of homogeneous coordinates?