Eigenvectors and eigenvalues

Definition. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{x}$ is a nonzero vector such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ, then λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is an eigenvector associated to the eigenvalue λ.

Basic facts about eigenvectors

1. Any nonzero scalar multiple of an eigenvector is another eigenvector associated to the same eigenvalue.
2. The equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}.$$

   If λ is an eigenvalue for $\mathbf{A}$, then Nul $(\mathbf{A} - \lambda\mathbf{I})$ is the eigenspace associated to λ.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The numbers $\lambda = 1$ and $\lambda = 2$ are eigenvalues of $\mathbf{A}$. What are the corresponding eigenspaces?

The $\lambda = 1$ eigenspace:

The $\lambda = 2$ eigenspace:

Theorem. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

How do we find the eigenvalues?

Note: The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if the homogeneous system

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

By the Invertible Matrix Theorem,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is **not** invertible.

The number λ is an eigenvalue for the matrix $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Example. Let
$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$

Example. Let
$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & -2 \\ -2 & 2 & -2 & 0 \\ -2 & 5 & 4 & -4 \\ 3 & 6 & -6 & -6 \end{bmatrix}.$$