

The characteristic polynomial and characteristic equation

Last class we saw that the number  $\lambda$  is an eigenvalue for the matrix  $\mathbf{A}$  if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Now I want to use the computer to examine the characteristic polynomial for various matrices.

**Example.** Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

**Example.** Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$

**Example.** Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

**Example.** Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 & -2 \\ -2 & 2 & -2 & 0 \\ -2 & 5 & 4 & -4 \\ 3 & 6 & -6 & -6 \end{bmatrix}.$$

**Theorem.** If a matrix is upper/lower triangular, then its eigenvalues are the entries along the main diagonal.

In theory, any polynomial  $p(\lambda)$  can be factored into irreducible linear and quadratic factors using real numbers. For example, consider the polynomial

$$p(\lambda) = \lambda^9 + 8\lambda^8 + 36\lambda^7 + 94\lambda^6 + 143\lambda^5 + 98\lambda^4 - 48\lambda^3 - 160\lambda^2 - 132\lambda - 40.$$

This polynomial factors into

$$p(\lambda) = (\lambda^2 + 2\lambda + 2)^2(\lambda^2 + 3\lambda + 10)^2(\lambda + 1)^2(\lambda - 1).$$

The **algebraic multiplicity** of an eigenvalue  $\lambda_0$  is the number of times that the factor  $(\lambda - \lambda_0)$  appears in the factorization of the characteristic polynomial  $p(\lambda)$ . The **geometric multiplicity** of  $\lambda_0$  is the dimension of its eigenspace.

**Theorem.** The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.

**Example.** Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

The diagonalization problem

“Diagonalizing” a matrix has many applications. One is geometric.

**Definition.** Two square matrices  $\mathbf{A}$  and  $\mathbf{B}$  are similar if there exists an invertible square matrix  $\mathbf{P}$  such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

**Note:** We have already done two exercises related to similarity—Section 2.2 #18 and Section 3.2 #34.

**Theorem.** Suppose that  $\mathbf{A}$  and  $\mathbf{B}$  are similar matrices.

1. Then they have the same characteristic polynomial and consequently the same eigenvalues.
2. They have the same geometric multiplicities for each eigenvalue.