The diagonalization problem

A matrix $\mathbf{A}$ is diagonalizable if there exists a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

"Diagonalizing" a matrix has many applications. One is geometric.

Diagonalizing a matrix is a special case of the similarity problem.

Definition. Two square matrices $\mathbf{A}$ and $\mathbf{B}$ are similar if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Note: We have already done two exercises related to similarity—Section 2.2 #18 and Section 3.2 #34.

Theorem. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are similar matrices.

1. Then they have the same characteristic polynomial and consequently the same eigenvalues.

2. They have the same geometric multiplicities for each eigenvalue.

What does this theorem say about matrices that can be diagonalized? For example, can

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

be diagonalized?

Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose $\mathbf{A}$has n linearly independent eigenvectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

and let λ_i be the eigenvalue that is associated to $\mathbf{v}_i$.

Note: The λ_i need not be distinct.

Then we can diagonalize $\mathbf{A}$ using the matrix

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \quad \ldots \quad & \mathbf{v}_n \end{array} \right].$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$