More on the diagonalization problem

Example. Consider the diagonal matrix that I made up during class on Friday

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

We know that it has three eigenvalues: $\lambda = 2$, 4, and 7.

The $\lambda = 2$ eigenspace is $\text{Span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}$.

The $\lambda = 4$ eigenspace is $\text{Span}\{\mathbf{e}_4\}$.

The $\lambda = 7$ eigenspace is $\text{Span}\{\mathbf{e}_2, \mathbf{e}_6\}$.

In general, a matrix $\mathbf{A}$ is diagonalizable if there exists a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Assuming that a matrix $\mathbf{A}$ can be diagonalized, what must $\mathbf{A}$ and $\mathbf{D}$ have in common?

Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose $\mathbf{A}$ has n linearly independent eigenvectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

and let λ_i be the eigenvalue that is associated to $\mathbf{v}_i$.

Note: The λ_i need not be distinct.

Then we can diagonalize $\mathbf{A}$ using the matrix

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \quad\ldots\quad & \mathbf{v}_n \end{array}\right].$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Example. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Now let's return to the unusual matrix that is in the animation.

Example. Consider the matrix

$$\mathbf{B} = \begin{bmatrix} \frac{31}{45} & \frac{19}{45} \\ -\frac{19}{90} & \frac{119}{90} \end{bmatrix}.$$