

The dot product

Definition. Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in \mathbb{R}^n , then

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n.$$

Example. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \pi \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -3\sqrt{2} \\ 2 \end{bmatrix}.$$

Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} be an $n \times p$ matrix. Write \mathbf{A} and \mathbf{B} as

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{array} \right] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_p \end{array} \right].$$

Let's interpret the product \mathbf{AB} in terms of the dot product.

Theorem 1. Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in \mathbb{R}^n , and let c be a scalar. Then

- (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- (c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- (d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, and
 $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Definition. The length (or norm) of a vector \mathbf{v} in \mathbb{R}^n is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

If we think of two vectors \mathbf{u} and \mathbf{v} as points in \mathbb{R}^n , then we define the distance between \mathbf{u} and \mathbf{v} as

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

What about angles? Let's start with right angles.

Definition. Two vectors \mathbf{u} and \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

We can use the Law of Cosines to derive a more exact relationship between angles and the dot product.

Orthogonal complements

Given a subspace S of \mathbb{R}^n , we can consider the set of all vectors that are orthogonal to all vectors in S . For example, a plane through the origin in \mathbb{R}^3 can be described by one homogeneous linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Definition. Given a subspace S of \mathbb{R}^n , its orthogonal complement S^\perp is the set

$$\{\mathbf{v} \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in S\}.$$

Examples.

1. The orthogonal complement of a line through the origin in \mathbb{R}^3 is a plane through the origin.
2. The orthogonal complement of a plane through the origin in \mathbb{R}^3 is a line through the origin.
3. Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}.$$

What can you say about a vector \mathbf{v} in \mathbb{R}^n that is orthogonal to all of the rows of \mathbf{A} ?

Theorem. Let S be a subspace of \mathbb{R}^n and let S^\perp be its orthogonal complement. Then

1. S^\perp is a subspace of \mathbb{R}^n ,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector \mathbf{v} in \mathbb{R}^n can be written uniquely as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

where \mathbf{v}_1 is in S and \mathbf{v}_2 is in S^\perp .