Orthogonal complements

Given a subspace S of $\mathbb{R}^n$, we can consider the set of all vectors that are orthogonal to all vectors in S. For example, a plane through the origin in $\mathbb{R}^3$ can be described by one homogeneous linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Definition. Given a subspace S of $\mathbb{R}^n$, its orthogonal complement $S^\perp$ is the set

$$\{\mathbf{v} \mid \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in S\}.$$

Examples.

1. The orthogonal complement of a line through the origin in $\mathbb{R}^3$ is a plane through the origin.

2. The orthogonal complement of a plane through the origin in $\mathbb{R}^3$ is a line through the origin.

3. Consider an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{bmatrix}.$$

   What can you say about a vector $\mathbf{v}$ in $\mathbb{R}^n$ that is orthogonal to all of the rows of $\mathbf{A}$?

Theorem. Let S be a subspace of $\mathbb{R}^n$ and let $S^\perp$ be its orthogonal complement. Then

1. $S^\perp$ is a subspace of $\mathbb{R}^n$,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as

   $$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

   where $\mathbf{v}_1$ is in S and $\mathbf{v}_2$ is in $S^\perp$.

Orthogonal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$.

Example 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

Theorem. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors.

1. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.
2. If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, then the weights c_i are given by

$$c_i = \frac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}.$$

Example. Consider the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in Example 1. Express

$$\mathbf{v}_4 = \begin{bmatrix} -1 \\ 12 \\ 44 \\ -37 \end{bmatrix}$$

as a linear combination of the orthogonal set.

Orthonormal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is orthonormal if it is orthogonal and $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can use matrices to express the fact that a set is orthonormal.