More on coordinates relative to a given basis

Last class we talked about how a basis produces a coordinate system on a vector space via the Unique Representation Theorem. In particular, recall the following example.

Example. Consider the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$. Then

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

because

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Change of coordinates matrix

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$, then the B-coordinates of a vector $\mathbf{x}$ are related to the standard coordinates by the equation

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\mathbf{x} = \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \mathbf{P}_B \, [\mathbf{x}]_B$$

where $\mathbf{P}_B$ is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \quad \ldots \quad & \mathbf{b}_n \end{array} \right].$$

Since $\mathbf{P}_B$ is invertible, we also have

$$[\mathbf{x}]_B = (\mathbf{P}_B)^{-1} \, \mathbf{x}.$$

Example. We can double check our computation of the B-coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$ relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

using these equations.

For any vector space V with basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, the B-coordinates define a nice linear transformation from V onto $\mathbb{R}^n$. The map is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

Theorem. The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps V onto $\mathbb{R}^n$.

Definition. A one-to-one linear transformation that maps V onto W is called an isomorphism.

From the vector space point of view, two isomorphic vector spaces have the same structure.

Example. For what n is $\mathbb{R}^n$ isomorphic to $\mathbb{P}_3$?

The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

Definition. A vector space V is finite dimensional if it contains a finite spanning set. Otherwise, V is said to be infinite dimensional.

Example. $\mathbb{R}^n$ is spanned by the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Therefore, it is finite dimensional.

Example. $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V.

Corollary. Any two bases of a finite-dimensional vector space V have the same number of elements.

Definition. The dimension of a finite-dimensional vector space V is the number of elements in any basis of V. This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$

2. Let P be the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

   Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in V with more than n vectors must be linearly dependent.

Theorem. If H is a subspace of V, then $\dim H \leq \dim V$. In fact, any basis of H can be expanded to to a basis of V.