More on the dimension of a vector space

Last class we saw that the number of elements of a basis of a finite-dimensional vector space does not depend on the particular choice of basis.

Theorem. Let V be a vector space. Any finite spanning set for V has at least as many elements as any linearly independent subset of V.

Corollary. Any two bases of a finite-dimensional vector space V have the same number of elements.

Definition. The dimension of a finite-dimensional vector space V is the number of elements in any basis of V. This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$
2. Let P be the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

   Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in V with more than n vectors must be linearly dependent.

Theorem. If H is a subspace of V, then $\dim H \leq \dim V$. In fact, any basis of H can be expanded to a basis of V.

Suppose that $\mathbf{A}$ is an $m \times n$ matrix. How can we determine the dimensions of Col $\mathbf{A}$ and Nul $\mathbf{A}$?

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

The rank of a matrix

Recall that the row space of an $m \times n$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$.

Theorem. If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then Row $\mathbf{A}$ = Row $\mathbf{B}$.

How do we find a basis for Row $\mathbf{A}$?

Example. Let's return to the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

For an $m \times n$ matrix $\mathbf{A}$, we saw that

$$\dim(\text{Col } \mathbf{A}) + \dim(\text{Nul } \mathbf{A}) = n.$$

How is the dimension of Row $\mathbf{A}$ related to these numbers?

Definition. The rank of an $m \times n$ matrix $\mathbf{A}$ is the dimension of its column space. This dimension also equals the dimension of Row $\mathbf{A}$.

Example. Suppose that a homogeneous linear system of 10 equations in 6 unknowns has two linearly independent solutions and all other solutions are linear combinations of these. Can the solution set be described with fewer equations? If so, how many?