

The rank of a matrix

Definition. The rank of an $m \times n$ matrix \mathbf{A} is the dimension of its column space. This dimension also equals the dimension of Row \mathbf{A} .

Example. Suppose that a homogeneous linear system of 10 equations in 6 unknowns has two linearly independent solutions and all other solutions are linear combinations of these. Can the solution set be described with fewer equations? If so, how many?

More equivalent conditions can be added to the Invertible Matrix Theorem.

Theorem. Let \mathbf{A} be an $n \times n$ matrix. Then the following seven statements are equivalent.

1. The matrix \mathbf{A} is invertible.
2. The columns of \mathbf{A} form a basis of \mathbb{R}^n .
3. $\text{Col } \mathbf{A} = \mathbb{R}^n$
4. $\dim(\text{Col } \mathbf{A}) = n$
5. $\text{rank } \mathbf{A} = n$
6. $\text{Nul } \mathbf{A} = \{\mathbf{0}\}$
7. $\dim(\text{Nul } \mathbf{A}) = 0$

Since \mathbf{A} is invertible if and only if \mathbf{A}^T is invertible, any statement regarding the columns of \mathbf{A} in the Invertible Matrix Theorem can be replaced by a statement regarding the rows of \mathbf{A} .

Eigenvectors and eigenvalues

Eigenvalues and eigenvectors are special numbers and vectors associated to certain matrices. They are useful in many applications. In particular, we will use them to help us understand how a linear transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ transforms \mathbb{R}^n .

Example. Consider the matrix

$$\mathbf{B} = \begin{bmatrix} \frac{31}{45} & \frac{19}{45} \\ -\frac{19}{90} & \frac{119}{90} \end{bmatrix}.$$

There is a Quicktime movie on the web site that illustrates how B and its powers B^2, \dots, B^7 transform the unit square in \mathbb{R}^2 .

Definition. Let \mathbf{A} be an $n \times n$ matrix. If \mathbf{x} is a nonzero vector such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some scalar λ , then λ is an eigenvalue of \mathbf{A} and \mathbf{x} is an eigenvector associated to the eigenvalue λ .

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

There is a link on the web page to a java applet called **Eigen Engine**. It helps us visualize a few special cases.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}.$$