Symmetric matrices

Symmetric matrices arise frequently in applications. Moreover, they have a particularly nice structure that can often be used to solve the problem at hand. Today we discuss that structure.

Definition. A matrix $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$.

Example. Consider

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

With the aid of *Mathematica*, we see that $\mathbf{A}$ has three distinct real eigenvalues, $\lambda = 7$, $\lambda = 4$, and $\lambda = 1$. We also have three eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

We can diagonalize $\mathbf{A}$ using an *orthogonal* matrix $\mathbf{P}$.

Theorem. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors associated to distinct eigenvalues of a symmetric matrix $\mathbf{A}$. Then $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Example. Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -2 & -2 & -1 \end{bmatrix}.$$

Theorem. (Spectral Theorem for symmetric matrices) If $\mathbf{A}$ is an $n \times n$ symmetric matrix, then

1. $\mathbf{A}$ has n real eigenvalues (counted with multiplicity),
2. the geometric multiplicity of each eigenvalue is the same as its algebraic multiplicity, and
3. distinct eigenspaces are mutually orthogonal.

Consequently, any symmetric matrix is orthogonally diagonalizable.