Example. Consider the system of equations

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 6 \\ x_1 + x_2 &= 3 \\ x_1 + x_3 &= 1. \end{aligned}$$

The augmented matrix associated to this system is

$$\begin{bmatrix} 2 & 1 & -1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Now we use row operations to solve the system of equations:

1. Flip rows R_1 and R_3:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 6 \end{bmatrix}$$

2. Replace row R_2 with $R_2 - R_1$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & -1 & 6 \end{bmatrix}$$

3. Replace row R_3 with $R_3 - 2R_1$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -3 & 4 \end{bmatrix}$$

4. Replace row R_3 by $R_3 - R_2$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

   At this point we know that the (original) system is consistent and that it has exactly one solution.

We can determine this solution with three more row operations:

5. Replace row R_3 by $(-\frac{1}{2})R_3$:
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

6. Replace row R_2 by $R_2 + R_3$:
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

7. Replace row R_1 by $R_1 - R_3$:
$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, the original system has the unique solution $(x_1, x_2, x_3) = (2, 1, -1)$, which we can check by calculating
$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2(2) + (1) - (-1) = 6 \\ x_1 + x_2 &= (2) + (1) \quad = 3 \\ x_1 + x_3 &= (2) + (-1) = 1. \end{aligned}$$

Row echelon form

A matrix is in **row echelon form** (REF) if it satisfies the following three properties:

1. All nonzero rows are above the zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form and it satisfies the additional two conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Examples. Consider the following three matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 0 & 3 & 7 & 2 \\ \pi & -1 & 4 & 2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 4 & 6 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem. Each matrix is row equivalent to a unique reduced row echelon matrix.