The vector space $\mathbb{R}^n$

Definition. The vector space $\mathbb{R}^n$ is the set of all n-tuples of real numbers. That is, $\mathbb{R}^n$ is the set of all possible $n \times 1$ "column vectors" of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where x_k is a real number for $k = 1, 2, \ldots, n$.

Vector addition: Given two vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

the vector sum $\mathbf{v} + \mathbf{w}$ is the vector

$$\begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

Vector addition can be visualized using the parallelogram rule.

Scalar multiplication: Given a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

and a real number (a "scalar") r, then

$$r\mathbf{v} = \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}.$$

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

- $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ commutative property
- $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ associative property
- $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ zero vector
- $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$ $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
- $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ distributive property
- $(c+d)\mathbf{u}+c\mathbf{u}+d\mathbf{u}$
- $c(d\mathbf{u})=(cd)\mathbf{u}$
- $1\mathbf{u}=\mathbf{u}$

Example. The set of all points (x_1, x_2, x_3) in $\mathbb{R}^3$ that satisfy the equation

$$x_1+x_2+x_3=0$$

is a plane. How can we describe this plane using the vector operations?

Definition. Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$ in $\mathbb{R}^n$ and some choice of real numbers r_1, r_2, ..., r_p, then the vector

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\ldots+r_p\mathbf{v}_p$$

is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$. The numbers r_1, r_2, ..., r_p are called the weights of the linear combination.

Examples.

Important question: Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$ as well as a vector $\mathbf{b}$, is $\mathbf{b}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$?

Example. Given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b}_1 = \begin{bmatrix} -3 \\ -2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}.$$

Is either $\mathbf{b}_1$ or $\mathbf{b}_2$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$?

Definition. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in $\mathbb{R}^n$. The set of all possible linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ is called the

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}.$$

Note:

1. Every scalar multiple of each $\mathbf{v}_k$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
2. The zero vector is always in the span of any set of vectors.
3. The $\text{span}\{\mathbf{v}_1\}$ is the set of all scalar multiples of $\mathbf{v}_1$.

Example. The set of all points (x_1, x_2, x_3) in $\mathbb{R}^3$ that satisfy the equation

$$x_1 + x_2 + x_3 = 0$$

is a plane. How can we describe this plane using the vector operations?

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}.$$

For what values of x_3 is the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ x_3 \end{bmatrix}$$

a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? What does this mean geometrically?