

Given the matrix-vector product \mathbf{Ax} , we can now establish the following theorem.

Theorem. Let \mathbf{A} be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each \mathbf{b} in \mathbb{R}^m , the equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution.
2. The columns of \mathbf{A} span \mathbb{R}^m .
3. The matrix \mathbf{A} has a pivot position in every row.

Warning: In this theorem, \mathbf{A} is a *coefficient* matrix. The three statements are not equivalent if \mathbf{A} is an augmented matrix.

Observation. Note that the k th entry in \mathbf{Ax} is

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n.$$

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n$$

is called the **dot product** of $[a_{k1} \ a_{k2} \ \dots \ a_{kn}]$ and the vector \mathbf{x} .

Theorem. Let \mathbf{A} be an $m \times n$ matrix. Then the matrix-vector product \mathbf{Ax} is “linear” in \mathbf{x} . That is,

1. $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$ for all \mathbf{u} and \mathbf{v} in \mathbb{R}^n , and
2. $\mathbf{A}(c\mathbf{u}) = c\mathbf{Au}$ for all \mathbf{u} in \mathbb{R}^n and all c in \mathbb{R} .

Solution sets of systems of linear equations

Definition. Consider a linear system $\mathbf{Ax} = \mathbf{b}$. We say that it is *homogeneous* if $\mathbf{b} = \mathbf{0}$ and *nonhomogeneous* otherwise.

The homogeneous case $\mathbf{Ax} = \mathbf{0}$

Observation. Note that every homogeneous system is consistent. The solution $\mathbf{x} = \mathbf{0}$ is called the *trivial* solution. All other solutions are said to be nontrivial.

Theorem. If \mathbf{v}_1 and \mathbf{v}_2 are two solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then any linear combination of \mathbf{v}_1 and \mathbf{v}_2 is also a solution.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 0 & -1 & -2 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Calculate the solution set for $\mathbf{Ax} = \mathbf{0}$.

The nonhomogeneous case $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$

Theorem. Let \mathbf{p} be one solution to the nonhomogeneous equation and let S be the set of all solutions to the associated homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ consists of all vectors in the set

$$\mathbf{p} + S.$$

Example. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ whose augmented matrix is

$$\left[\mathbf{A} \mid \mathbf{b} \right] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 9 \end{bmatrix}.$$