Solution sets of nonhomogeneous systems of linear equations

Theorem. Let $\mathbf{p}$ be one solution to the nonhomogeneous equation and let S be the set of all solutions to the associated homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ consists of all vectors in the set $\mathbf{p} + S$.

Example. Consider the linear system $\mathbf{Ax} = \mathbf{b}$ whose augmented matrix is

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 9 \end{array}\right].$$

Linear independence

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We have already seen that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane P in $\mathbb{R}^3$ given by the equation

$$x_1 + x_2 + x_3 = 0.$$

What happens to the span if we add a third vector $\mathbf{v}_3$ to the set of vectors generating the span? In other words, what is $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Definition. A (linear) dependence relation among a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is an equation of the form

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some vector $\mathbf{v}_i \in \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$.

Example.

$$2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition. If there exists a dependence relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$$

among a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$, then we say that the set is *linearly dependent.* A set is *linearly independent* if it is not linearly dependent.

Matrix characterization

A dependence relation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$ can be rewritten as the matrix equation $\mathbf{Ar} = \mathbf{0}$ where

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \end{array}\right] \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}.$$

Therefore, a dependence relation among the vectors $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_k$ is the same as a nontrivial solution to $\mathbf{Ar} = \mathbf{0}$.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We can determine that these three vectors are linearly independent by considering the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Example. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent? Why?

1. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ \pi \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$