More on linear independence

Definition. If there exists a dependence relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$$

among a set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$, then we say that the set is *linearly dependent*. A set is *linearly independent* if it is not linearly dependent.

Matrix characterization

A dependence relation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_k\mathbf{v}_k = \mathbf{0}$ can be rewritten as the matrix equation $\mathbf{A}\mathbf{r} = \mathbf{0}$ where

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \end{array}\right] \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_k \end{bmatrix}.$$

Therefore, a dependence relation among the vectors $\mathbf{v}_1,\ \ldots,\ \mathbf{v}_k$ is the same as a nontrivial solution to $\mathbf{A}\mathbf{r} = \mathbf{0}$.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We can determine that these three vectors are linearly independent by considering the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Example. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent? Why?

1. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

2. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$

3. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ \pi \end{bmatrix} \right\}$

4. $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 13 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

5. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \right\}$

Theorem. A nonzero set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors is linearly dependent if and only if, for some index j, the vector $\mathbf{v}_j$ is a linear combination of the vectors $\mathbf{v}_1$, ..., $\mathbf{v}_{j-1}$.

Theorem. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

Linear transformations

In order to understand the definition of a linear transformation, let's start with some examples.

Examples: functions $f : \mathbb{R} \to \mathbb{R}$

1. $f_1(x) = 2x$
2. $f_2(x) = 2x + 1$
3. $f_3(x) = x^2$
4. $f_4(x) = \cos x$

Examples: functions $g : \mathbb{R}^2 \to \mathbb{R}^2$

1. $g_1(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$
2. $g_2(x_1, x_2) = (\cos(x_1 + x_2), x_1 + x_2^2)$

Examples: functions h defined on $\mathbb{R}^3$

1. $h_1(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2 + x_3)$
2. $h_2(x_1, x_2, x_3) = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Definition. Given a function (transformation) $T : \mathbb{R}^n \to \mathbb{R}^m$, we say that T is *linear* if

1. $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, and
2. $T(r\mathbf{v}) = rT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$and all r in $\mathbb{R}$.

Terminology: Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$:

- $\mathbb{R}^n$ is the *domain* of T.
- Our textbook says that $\mathbb{R}^m$ is the *codomain* of T.
- The *image* or *range* of T is the set of vectors

$$\{\mathbf{w} \in \mathbb{R}^m \mid T(\mathbf{v}) = \mathbf{w} \text{ for some } \mathbf{v} \in \mathbb{R}^n\}.$$

Basic facts about linear transformations T

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(r_1\mathbf{v}_1 + \ldots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + \ldots + r_kT(\mathbf{v}_k)$

Which of the functions given above are linear?

Important class of examples: Given an $m \times n$ matrix $\mathbf{A}$, then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by the equation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$