More on linear transformations

Last class I mentioned that the functions

$$f_1(x) = 2x$$

$$g_1(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

$$h_1(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2 + x_3)$$

$$h_2(x_1, x_2, x_3) = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

are all linear transformations, but I did not explain why. We can verify the two conditions of linearity for each example directly, but there is an easier way to see that these transformations are linear.

Important class of examples: Given an $m \times n$ matrix $\mathbf{A}$, then we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by the equation

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

We know that T is a linear transformation because the matrix-vector product satisfies the necessary conditions.

Example. Let

$$\mathbf{G} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then

$$\mathbf{G}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

There is a link on the course web page to a java program called `Matrix Machine` that lets you investigate the mapping properties of various 2×2 matrix transformations. Try the following three matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Theorem. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $m \times n$ matrix. This matrix $\mathbf{A}$ is called the *standard matrix representation* of T.

Why? Let's make two observations using the "standard basis" of $\mathbb{R}^n$.

Definition. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \ldots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

1. If we know the images $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ for all of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then we can calculate $T(\mathbf{x})$ for any $\mathbf{x}$ in $\mathbb{R}^n$.

2. If

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{array} \right],$$

then

$$\mathbf{A}\mathbf{x} = x_1 T(\mathbf{e}_1) + \ldots + x_n T(\mathbf{e}_n).$$

Example. What is the matrix representation of the linear transformation that rotates $\mathbb{R}^2$ by 45° ($\pi/4$ radians) around the origin?

Examples of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$

1. rotations

2. reflections

3. contractions and expansions

4. shears

5. projections