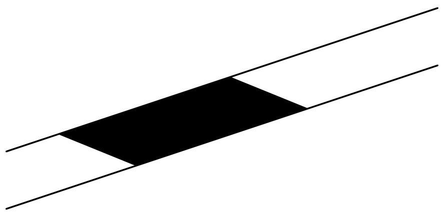
More on the Geometry of the Determinant

Last class we observed that the areas of the following two parallelograms are equal.

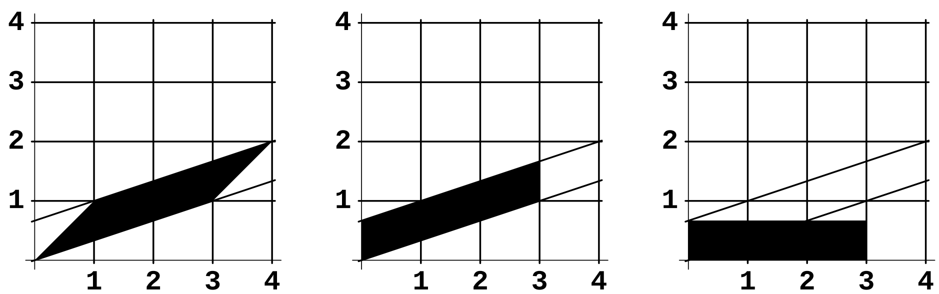


Consider a parallelogram $P$ in $\mathbb{R}^2$ formed by two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$.

**Example.** Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let's calculate the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ in an unusual way.



**Summary:** $2 \times 2$ Matrices and Area in $\mathbb{R}^2$

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is

$$\left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|.$$

Volume in $\mathbb{R}^3$

The same basic geometric argument that works in $\mathbb{R}^2$ also works in $\mathbb{R}^3$. Given three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$, consider the parallelopiped that they determine.

How do column operations affect the volume of the corresponding parallelopipeds?

**Summary:** $3 \times 3$ Matrices and Volume in $\mathbb{R}^3$

Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then the volume of the parallelopiped formed by **a**, **b**, and **c** is

$$\left| \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right|.$$

**Example.** Consider the parallelopiped generated by

$$\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix},$$

Determinants and linear transformations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a matrix $\mathbf{A}$ such that

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}.$$

What is the significance of $\det \mathbf{A}$ in this situation?

Consider the case where $n = 2$ and start with a parallelogram $P$ determined by two vectors $\mathbf{u}$ and $\mathbf{v}$.

**Summary:** Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, then $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some $2 \times 2$ matrix $\mathbf{A}$. If $P$ is a parallelogram determined by the two vectors $\mathbf{u}$ and $\mathbf{v}$, then $T(P)$ is also a parallelogram, and

$$\text{area of } T(P) = (\det A)(\text{area of } P).$$

For $n = 3$, the same conclusion holds if the concept of area is replaced by that of volume.

Also, there is nothing special about parallelograms in this discussion. We could just as well start with a region such as a disk.

For more details, see pp. 208–209 of our text.

If you have studied multivariable calculus, you know that there is a change of variables formula that is used to convert multiple integrals from one set of coordinates to another. That formula involves the determinant of the Jacobian matrix (see Stewart *Calculus: Concepts and Contexts*, Section 12.9). The area conversion formula mentioned here is a special case of that more general formula.