A little more matrix algebra

The properties of matrix multiplication are somewhat different from the properties of regular multiplication.

Three warnings.

1. $\mathbf{AB}$ does not always equal $\mathbf{BA}$.
2. $\mathbf{AB} = \mathbf{AC}$ does not necessarily imply that $\mathbf{B} = \mathbf{C}$.
3. $\mathbf{AB} = \mathbf{0}$ does not necessarily imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

We will occasionally need to use the transpose of a matrix.

Definition. Given an $m \times n$ matrix $\mathbf{A}$, its transpose $\mathbf{A}^T$ is the $n \times m$ matrix such that

$$(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}.$$

Example. Consider

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6\pi \end{bmatrix}.$$

Theorem 3. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices whose sizes are appropriate for the following sums and products. Then

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
3. $(r\mathbf{A})^T = r\mathbf{A}^T$
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Matrix inverses

Invertible square matrices are more well behaved than arbitrary square matrices.

Definition. Let $\mathbf{A}$ be a square matrix for which there exists a square matrix $\mathbf{B}$ such that either

1. $\mathbf{AB} = \mathbf{I}$ or
2. $\mathbf{BA} = \mathbf{I}$.

Then we say that $\mathbf{A}$ is *invertible* and that $\mathbf{B}$ is the *inverse* of $\mathbf{A}$.

Examples. Consider

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

There is a simple formula for the inverse of a 2×2 matrix.

Theorem 4. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $\mathbf{A}$ is not invertible.

Computing inverses

Here are some basic properties of inverses.

Theorem 6.

1. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

2. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then $\mathbf{AB}$ is invertible. Moreover, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

3. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^T$ is invertible, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Elementary matrices

Definition. An *elementary* matrix is a matrix that is obtained from the identity matrix by applying exactly one elementary row operation.

There are three types of elementary row operations—one for each type of row operation.

What happens to a matrix if we multiply it by an elementary matrix?

Example.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algorithm for computing $\mathbf{A}^{-1}$

Form the augmented matrix

$$[\ \mathbf{A}\mid\mathbf{I}\ ].$$

Row reduce this matrix so that the left half becomes the identity matrix. At that point, the right half is $\mathbf{A}^{-1}$.