More on subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Examples. Recall that the line $x_2 = 3x_1$ is a subspace of $\mathbb{R}^2$ but the line $x_2 = x_1 + 1$ is not.

Example. Let $\mathbb{P}$ represent the vector space of all polynomial functions as discussed last class. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is S a subspace of $\mathbb{P}$?

Theorem. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Example. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of V are subspaces of V?

1. The set of all constant functions.

2. The set of all functions f such that $f(2) = 1$.

3. The set of all functions f such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.

Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $\mathbf{A}$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ represent the columns of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{c}_1 & \mathbf{c}_2 & \quad \cdots \quad & \mathbf{c}_n \end{array} \right].$$

These column vectors are vectors in $\mathbb{R}^m$.

Let $\mathbf{r}_1, \ldots, \mathbf{r}_m$ represent the rows of $\mathbf{A}$. That is,

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_m \end{array} \right].$$

These row vectors are vectors in $\mathbb{R}^n$.

The column space of A. The column space of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$. We write

$$\text{Col } \mathbf{A} = \text{Span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}.$$

The row space of A. The row space of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. We write

$$\text{Row } \mathbf{A} = \text{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

The null space of A. The null space of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

The null space of $\mathbf{A}$ is denoted by Nul $\mathbf{A}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. The the column space of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$, and the null space and the row space of $\mathbf{A}$ are subspaces of $\mathbb{R}^n$.

Application. Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 2 & -4 & 0 & 8 & 1 \end{bmatrix}.$$

Express the null space of $\mathbf{A}$ as the span of as few vectors as possible.

The consistency of a system of linear equations can be viewed as a statement about the column space of the coefficient matrix.

Fact. The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is an element of the column space of $\mathbf{A}$.