

More on subspaces related to matrices

Recall that the null space of an $m \times n$ -matrix is a subspace of \mathbb{R}^n .

Application. Any plane through the origin in \mathbb{R}^3 is a subspace of \mathbb{R}^3 .

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 2 & -4 & 0 & 8 & 1 \end{bmatrix}.$$

Express the null space of \mathbf{A} as the span of as few vectors as possible.

The consistency of a system of linear equations can be viewed as a statement about the column space of the coefficient matrix.

Fact. The linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if \mathbf{b} is an element of the column space of \mathbf{A} .

Here is how Lay (p. 232) contrasts $\text{Nul } \mathbf{A}$ and $\text{Col } \mathbf{A}$ for an $m \times n$ matrix \mathbf{A} :

Nul \mathbf{A}	Col \mathbf{A}
1. Nul \mathbf{A} is a subspace of \mathbb{R}^n .	1. Col \mathbf{A} is a subspace of \mathbb{R}^m .
2. Nul \mathbf{A} is implicitly defined; that is, you are given only a condition ($\mathbf{Ax} = \mathbf{0}$) that vectors in Nul \mathbf{A} must satisfy.	2. Col \mathbf{A} is explicitly defined; that is, you are told how to build vectors in Col \mathbf{A} .
3. It takes time to find vectors in Nul \mathbf{A} . Row operations on $[\mathbf{A} \ \mathbf{0}]$ are required.	3. It is easy to find vectors in Col \mathbf{A} . The columns of \mathbf{A} are displayed; others are formed from them.
4. There is no obvious relation between Nul \mathbf{A} and the entries in \mathbf{A} .	4. There is an obvious relation between Col \mathbf{A} and the entries in \mathbf{A} , since each column of \mathbf{A} is in Col \mathbf{A} .
5. A typical vector \mathbf{v} in Nul \mathbf{A} has the property that $\mathbf{Av} = \mathbf{0}$.	5. A typical vector \mathbf{v} in Col \mathbf{A} has the property that the equation $\mathbf{Ax} = \mathbf{v}$ is consistent.
6. Given a specific vector \mathbf{v} , it is easy to tell if \mathbf{v} is in Nul \mathbf{A} . Just compute \mathbf{Av} .	6. Given a specific vector \mathbf{v} , it may take time to tell if \mathbf{v} is in Col \mathbf{A} . Row operations on $[\mathbf{A} \ \mathbf{v}]$ are required.
7. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the equation $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.	7. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the equation $\mathbf{Ax} = \mathbf{b}$ has a solution for every \mathbf{b} in \mathbb{R}^m .
8. Nul $\mathbf{A} = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.	8. Col $\mathbf{A} = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ maps \mathbb{R}^n onto \mathbb{R}^m .

Item 8 in both lists suggest two subspaces that are intimately connected with any linear transformation from one vector space to another.

Definition. A transformation $L : V_1 \rightarrow V_2$ from a vector space V_1 to a vector space V_2 is linear if

1. $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ for all vectors \mathbf{v}_1 and \mathbf{v}_2 in V_1 , and
2. $L(r\mathbf{v}) = rL(\mathbf{v})$ for all \mathbf{v} in V_1 and all r in \mathbb{R} .

Example. Let V_1 be the vector space of all continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and let V_2 be the vector space of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$. The operation of differentiation is a linear transformation from V_1 to V_2 . That is, the transformation $D : V_1 \rightarrow V_2$ given by

$$D(f) = f'$$

is a linear transformation.

Associated to any linear transformation are two important subspaces.

Definition. The kernel of $L : V_1 \rightarrow V_2$ is the subset of V_1 given by

$$\{\mathbf{v}_1 \mid L(\mathbf{v}_1) = \mathbf{0}\}.$$

The range of L is the subset of V_2 given by

$$\{\mathbf{v}_2 \mid L(\mathbf{v}_1) = \mathbf{v}_2 \text{ for some } \mathbf{v}_1 \text{ in } V_1\}.$$

Fact. Both the kernel and the range of a linear transformation are subspaces. The kernel is a subspace of V_1 , and the range is a subspace of V_2 .

For a matrix transformation from \mathbb{R}^n to \mathbb{R}^m determined by the matrix \mathbf{A} , its range is $\text{Col } \mathbf{A}$, and its kernel is $\text{Nul } \mathbf{A}$.

Example. What are the kernel and range of the transformation $p : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ determined by the matrix

$$\frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} ?$$

(This example was first introduced on September 25.)

Example. What are the kernel and the range of the differentiation transformation D mentioned above?