Coordinates relative to a basis

A basis for a vector space produces a coordinate system for that space.

**Theorem.** (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $V$. Then every vector $\mathbf{v}$ in $V$ can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

The scalars $c_1, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis $B$.

**Example.** Consider the spanning set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ for the vector space $\mathbb{P}_3$. There are infinitely many ways to write a given element of $\mathbb{P}_3$ as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$

It can also be written as

$$2(x^3 + 1) + (-1)x + (-1)(x^2 - x) + (-\tfrac{1}{2})(4).$$

Because this spanning set is not linearly independent, there is no *unique* representation of $2x^3 - x^2$ as a linear combination of the vectors.

Last class we produced a basis of $\mathbb{P}_3$ from this spanning set using the casting-out procedure. The basis is $\{x^3 + 1, x, x^2, 4\}$. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Why are coordinates relative to a given basis unique?

The same vector has different coordinates relative to different bases.

**Example.** Consider the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$. What are its coordinates relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and what are its coordinates relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}?$$

**Notation.** Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in $\mathbb{R}^n$. This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

**Example.** What is the coordinate vector for $2x^3 - x^2$ relative to the basis $B = \{x^3 + 1, x, x^2, 4\}$ of $\mathbb{P}_3$?

Change of coordinates matrix

If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$, then the $B$-coordinates of a vector $\mathbf{x}$ are related to the standard coordinates by the equation

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\begin{aligned} \mathbf{x} &= \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \mathbf{P}_B \, [\mathbf{x}]_B \end{aligned}$$

where $\mathbf{P}_B$ is the matrix

$$\mathbf{P}_B = \left[ \begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array} \right].$$

Since $\mathbf{P}_B$ is invertible, we also have $[\mathbf{x}]_B = (\mathbf{P}_B)^{-1}\,\mathbf{x}$.

**Example.** We can double check our computation of the $B$-coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in $\mathbb{R}^2$ relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

using these equations.

For any vector space $V$ with basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, the $B$-coordinates define a nice linear transformation from $V$ onto $\mathbb{R}^n$. The map is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

**Theorem.** The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps $V$ onto $\mathbb{R}^n$.

**Definition.** A one-to-one linear transformation that maps $V$ onto $W$ is called an isomorphism.

From the vector space point of view, two isomorphic vector spaces have the same structure.

**Example.** For what $n$ is $\mathbb{R}^n$ isomorphic to $\mathbb{P}_3$?