The Invertible Matrix Theorem

**Theorem.** Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following twelve statements are equivalent:

(a) $\mathbf{A}$ is an invertible matrix.

(b) $\mathbf{A}$ is row equivalent to the identity matrix.

(c) $\mathbf{A}$ has $n$ pivot positions

(d) The equation $\mathbf{Ax} = \mathbf{0}$ has no nontrivial solutions.

(e) The columns of $\mathbf{A}$ are linearly independent.

(f) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ is one-to-one.

(g) The equation $\mathbf{Ax} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.

(h) The columns of $\mathbf{A}$ span $\mathbb{R}^n$.

(i) The linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.

(k) There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}$.

(l) $\mathbf{A}^T$ is an invertible matrix.

Computer graphics

Homogeneous coordinates are useful when we want to do computer graphics with matrices.

**Definition.** A point $(x, y)$ in $\mathbb{R}^2$ can be represented by the point $(x, y, 1)$ in $\mathbb{R}^3$. The coordinates $(x, y, 1)$ are called the homogeneous coordinates of the point $(x, y)$.

Homogeneous coordinates are useful because translation in $\mathbb{R}^2$ can be represented by a linear transformation in $\mathbb{R}^3$.

**Fact 1.** A translation by $(h, k)$ in $\mathbb{R}^2$ can be obtained by matrix multiplication of homogeneous coordinates. That is,

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}.$$

**Fact 2.** Any linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ can be represented as a transformation of homogeneous coordinates by matrix multiplication. In particular, if the transformation is represented by the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the corresponding matrix for homogeneous coordinates is

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example.** What is the matrix that represents rotation by 45° in terms of homogeneous coordinates?