A little more on orthogonal complements

Definition. Given a subspace S of $\mathbb{R}^n$, its orthogonal complement $S^\perp$ is the set

$$\{\mathbf{v} \mid \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in S\}.$$

Theorem. Let S be a subspace of $\mathbb{R}^n$ and let $S^\perp$ be its orthogonal complement. Then

1. $S^\perp$ is a subspace of $\mathbb{R}^n$,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is in S and $\mathbf{v}_2$ is in $S^\perp$.

It helps to have a little more theory before we can verify properties 2, 3, and 4, but we can verify property 1 directly from the definition.

Orthogonal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set if $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ for all $i \neq j$.

Example 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$

Theorem. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors.

1. If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, then the weights c_i are given by $c_i = \dfrac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$.

2. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

Example. Using the orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in Example 1, apply this theorem to the vector

$$\mathbf{v} = \begin{bmatrix} -45 \\ -4 \\ 3 \\ 1 \end{bmatrix}.$$

Orthonormal sets

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is orthonormal if it is orthogonal and $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for all i.

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

We can use matrices to express the fact that a set is orthonormal.