Projection matrices

We discussed projection matrices briefly last class. In particular, we discussed the following theorem.

Theorem. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{U} = \left[\begin{array}{c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \quad \cdots \quad & \mathbf{u}_k \end{array} \right].$$

Then $\text{proj}_W \mathbf{v} = \mathbf{U}\mathbf{U}^T\mathbf{v}$.

The matrix $\mathbf{P} = \mathbf{U}\mathbf{U}^T$ is called the *projection matrix* for the subspace W. It does not depend on the choice of orthonormal basis.

Example. Let's compute the projection matrix $\mathbf{P}$ for orthogonal projection onto the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$.

What are the eigenvalues and eigenspaces of $\mathbf{P}$? (No computation required)

What if we do not start with an orthonormal basis of W?

Theorem. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be any basis for a subspace W of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \quad \ldots \quad & \mathbf{a}_k \end{array} \right].$$

Then the projection matrix for W is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

A proof of this fact is posted on the course web site.

Note that any projection matrix $\mathbf{P}$ satisfies the two properties

1. $\mathbf{P}^2 = \mathbf{P}$, and
2. $\mathbf{P}$ is symmetric.

It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace of $\mathbb{R}^n$.

Least squares approximation

Suppose we have data points $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_n, y_n)$ and we want to "fit" them to a line.

How do we find the equation $y = mx + b$ of the line?

Form the matrices

$$\mathbf{Y}_d = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}.$$

If all of the points are on the line $y = mx + b$, we have

$$\mathbf{Y}_d = \mathbf{X} \begin{bmatrix} b \\ m \end{bmatrix},$$

and we are done.

If not, we consider all $\mathbf{Y}$ of the form

$$\mathbf{Y} = \mathbf{X} \begin{bmatrix} b \\ m \end{bmatrix}$$

for all possible b and m, and we look for the one that is "closest" to $\mathbf{Y}_d$. In other words, we minimize

$$||\mathbf{Y}_d - \mathbf{X} \begin{bmatrix} b \\ m \end{bmatrix}||$$

as we vary b and m.

Note also that we sweep out the column space of $\mathbf{X}$ as we vary b and m. Therefore, the minimum is attained when

$$\mathbf{X}\begin{bmatrix} b \\ m \end{bmatrix}$$

is the projection of $\mathbf{Y}_d$ onto the column space of $\mathbf{X}$.

Using the formula given earlier for the projection matrix, we have

$$\mathbf{X}\begin{bmatrix} b \\ m \end{bmatrix} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}_d.$$

Although $\mathbf{X}$ is not square and therefore not invertible, it has rank 2, and consequently, the transformation induced by $\mathbf{X}$ is one-to-one. We can cancel $\mathbf{X}$ from the left on both sides, and we obtain

$$\begin{bmatrix} b \\ m \end{bmatrix} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}_d.$$

Example. Suppose that a company had profits of \$500,000 in year 1, \$1,000,000 in year 2, and \$2,000,000 in year 5. Model its profits with a least-squares linear model.

Example. Here are relative growth rates for the U.S. population from 1800 to 1990.

Year	U.S. Population	Rel Growth Rate
1800	5.3	0.03113
1810	7.2	0.02986
1820	9.6	0.02500
1830	12	0.03083
1840	17	0.03235
1850	23	0.03043
1860	31	0.02419
1870	38	0.02500
1880	50	0.02400
1890	62	0.02016
1900	75	0.01933
1910	91	0.01648
1920	105	0.01476
1930	122	0.01066
1940	131	0.01107
1950	151	0.01589
1960	179	0.01453
1970	203	0.01170
1980	226	0.01015
1990	249	0.01094

Here's a graph of these relative growth rates versus population: