More on quadratic forms

Last class we saw that the Spectral Theorem for symmetric matrices can be used to eliminate "mixed" terms in a quadratic form. In fact, the Spectral Theorem implies the Principal Axes Theorem for quadratic forms.

Today we examine a few consequences.

Quadratic forms are classified according to the signs of their values.

Definition. A quadratic form Q is:

1. positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
2. negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, and
3. indefinite if $Q(\mathbf{x})$ assumes both positive and negative values.

There are also definitions of positive/negative semidefinite quadratic forms given in the textbook.

Example. Suppose that the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ is positive definite. What does the graph of

$$x_3 = Q(x_1, x_2)$$

look like?

Theorem. Let $\mathbf{A}$ be a symmetric matrix. Then the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is:

1. positive definite if and only if all of the eigenvalues of $\mathbf{A}$ are positive.
2. negative definite if and only if all of the eigenvalues of $\mathbf{A}$ are negative.
3. indefinite if and only if $\mathbf{A}$ has both positive and negative eigenvalues.

I would end the semester by returning to a topic that we discussed early in the semester. On September 20, we discussed the structure of solution sets to nonhomogeneous linear equations. Here's a repeat of a theorem from that class:

Theorem. Let $\mathbf{p}$ be one solution to the nonhomogeneous equation $\mathbf{Ax} = \mathbf{b}$ and let S be the set of all solutions to the associated homogeneous equation

$$\mathbf{Ax} = \mathbf{0}.$$

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ consists of all vectors in the set $\mathbf{p} + S$.

In Exercise 23 of Section 6.3, we proved that there is a unique $\mathbf{p} \in \text{Row } \mathbf{A}$ that is a solution to a consistent nonhomogeneous system. Now I would like to describe this result using diagrams advocated by Professor Gilbert Strang at MIT.