Quadratic forms

A quadratic form on $\mathbb{R}^n$ is a purely quadratic expression in the variables x_1, x_2, $\ldots$, x_n. For example, the expression

$$5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$$

is a quadratic form on $\mathbb{R}^3$. Quadratic forms arise in numerous applications (see our textbook). They are also a rich source of examples in various areas of mathematics. The web site for this course has a link to a nice web page at the University of Minnesota that discusses quadric surfaces, which are based on quadratic forms.

Every quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ can be written as

$$Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $n \times n$ symmetric matrix.

Example. Write $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$ as $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ for some 3×3 symmetric matrix $\mathbf{A}$.

We can use the Spectral Theorem for symmetric matrices to eliminate "mixed" terms such as x_1x_2, x_1x_3, $\ldots$ by using a change of variables of the form

$$\mathbf{x} = \mathbf{P}\mathbf{y},$$

where $\mathbf{P}$ is an orthogonal matrix that diagonalizes $\mathbf{A}$.

If $\mathbf{x} = \mathbf{P}\mathbf{y}$ where $\mathbf{P}$ orthogonally diagonalizes $\mathbf{A}$, then

Example. In $\mathbb{R}^2$, consider the set $5x^2 + 6xy + 5y^2 = 8$. Sketch this curve.

Theorem. (Principal Axes Theorem) Let $\mathbf{A}$ be a symmetric matrix. Then there is an orthogonal change of variable $\mathbf{x} = \mathbf{P}\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ into a quadratic form $\mathbf{y}^T\mathbf{D}\mathbf{y}$, where $\mathbf{D}$ is a diagonal matrix. Consequently, the new quadratic form has no mixed terms, i.e., no terms of the form $y_i y_j$ for $i \neq j$.

Quadratic forms are classified according to the signs of their values.

Definition. A quadratic form Q is:

1. positive definite if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
2. negative definite if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, and
3. indefinite if $Q(\mathbf{x})$ assumes both positive and negative values.

There are also definitions of positive/negative semidefinite quadratic forms given in the textbook.

Example. Suppose that the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ is positive definite. What does the graph of

$$x_3 = Q(x_1, x_2)$$

look like?

Theorem. Let $\mathbf{A}$ be a symmetric matrix. Then the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is:

1. positive definite if and only if all of the eigenvalues of $\mathbf{A}$ are positive.
2. negative definite if and only if all of the eigenvalues of $\mathbf{A}$ are negative.
3. indefinite if and only if $\mathbf{A}$ has both positive and negative eigenvalues.