Some terminology related to RREF

Example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

It is row equivalent to the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

The matrix **B** determines the pivot positions and columns of the matrix **A**.

Theorem 2.

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. In other words, a linear system is consistent if and only if the RREF matrix that is row equivalent to the augmented matrix has no row of the form $[0\ 0\ \ldots\ 0\ b]$, where b is nonzero.
2. If a linear system is consistent, then the solution set contains either
   (a) a unique solution, when there are no free variables, or
   (b) infinitely many solutions, where there is at least one free variable.

The vector space $\mathbb{R}^n$

Definition. The vector space $\mathbb{R}^n$ is the set of all n-tuples of real numbers. That is, $\mathbb{R}^n$ is the set of all possible $n \times 1$ "column vectors" of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where x_k is a real number for $k = 1, 2, \ldots, n$.

Vector addition: Given two vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

the vector sum $\mathbf{v} + \mathbf{w}$ is the vector

$$\begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

Vector addition can be visualized using the parallelogram rule.

Scalar multiplication: Given a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

and a real number (a "scalar") r, then

$$r\mathbf{v} = \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}.$$

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

- $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ commutative property
- $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ associative property
- $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ zero vector
- $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$ $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
- $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ distributive property
- $(c+d)\mathbf{u}+c\mathbf{u}+d\mathbf{u}$
- $c(d\mathbf{u})=(cd)\mathbf{u}$
- $1\mathbf{u}=\mathbf{u}$

Example. The set of all points (x_1, x_2, x_3) in $\mathbb{R}^3$ that satisfy the equation

$$x_1+x_2+x_3=0$$

is a plane. How can we describe this plane using vector operations?

Definition. Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$ in $\mathbb{R}^n$ and some choice of real numbers r_1, r_2, ..., r_p, then the vector

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\ldots+r_p\mathbf{v}_p$$

is said to be a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$. The numbers r_1, r_2, ..., r_p are called the weights of the linear combination.

Examples.

Important question: Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ as well as a vector $\mathbf{b}$, is $\mathbf{b}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$?

Example. Given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b}_1 = \begin{bmatrix} -3 \\ -2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}.$$

Is either $\mathbf{b}_1$ or $\mathbf{b}_2$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$?