The span of a set of vectors in $\mathbb{R}^n$

Definition. Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and some choice of real numbers $r_1, r_2, \ldots, r_p$, then the vector

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \ldots + r_p\mathbf{v}_p$$

is said to be a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$. The numbers $r_1, r_2, \ldots, r_p$ are called the weights of the linear combination.

Important question: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ as well as a vector $\mathbf{b}$, is $\mathbf{b}$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$?

Example. Given

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b}_1 = \begin{bmatrix} -3 \\ -2 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}.$$

Is either $\mathbf{b}_1$ or $\mathbf{b}_2$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

Last class we observed that writing $\mathbf{b}_1$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ is the same as solving the linear system of equations

$$\begin{aligned} r_1 - r_2 + 2r_3 + r_4 &= -3 \\ r_1 - r_2 + 3r_3 + 2r_4 &= -2 \\ 2r_1 + 2r_2 + 3r_3 + r_4 &= 3 \\ r_3 + r_4 &= -1, \end{aligned}$$

where the weights r_1, r_2, r_3, and r_4 are the variables.

Definition. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in $\mathbb{R}^n$. The set of all possible linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ is called the

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}.$$

Note:

1. Every scalar multiple of each $\mathbf{v}_k$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
2. The zero vector is always in the span of any set of vectors.
3. The $\text{span}\{\mathbf{v}_1\}$ is the set of all scalar multiples of $\mathbf{v}_1$.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

What vectors are in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

Example. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}.$$

For what values of x_3 is the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ x_3 \end{bmatrix}$$

in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$? What does this result mean geometrically?

The matrix-vector product $\mathbf{Ax}$

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{x}$ be a vector in $\mathbb{R}^n$. We can define the product $\mathbf{Ax}$ as a linear combination of the vectors that come from the columns of $\mathbf{A}$.

Definition. Let $\mathbf{A}$ be an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \left[\begin{array}{c|c|c|c} & & & \\ \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_n \\ & & & \end{array} \right],$$

where $\mathbf{A}_k$ is the kth column of $\mathbf{A}$. Given

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

in $\mathbb{R}^n$, we define the matrix-vector product $\mathbf{Ax}$ to be the linear combination

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \ldots + x_n\mathbf{A}_n.$$

Note that $\mathbf{Ax}$ is a vector in $\mathbb{R}^m$.

Example.

$$\begin{bmatrix} 3 & -8 \\ -1 & 5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-4)(3) + (2)(-8) \\ (-4)(-1) + (2)(5) \\ (-4)(2) + (2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} -28 \\ 14 \\ -2 \end{bmatrix}$$

Remark. Given an $m \times n$ matrix $\mathbf{A}$ and $\mathbf{x} \in \mathbb{R}^n$, then the matrix equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\left[\begin{array}{c|c|c|c|c} \mathbf{A}_1 & \mathbf{A}_2 & \dots & \mathbf{A}_n & \mathbf{b} \end{array} \right] = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array} \right].$$

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has at least one solution.
2. The columns of $\mathbf{A}$ span $\mathbb{R}^m$.
3. The matrix $\mathbf{A}$ has a pivot position in every row.

Warning: In this theorem, $\mathbf{A}$ is a *coefficient* matrix. The three statements are not equivalent if $\mathbf{A}$ is an augmented matrix.

Observation. Note that the kth entry in $\mathbf{Ax}$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$$

is called the **dot product** of $[\, a_{k1} \quad a_{k2} \quad \ldots \quad a_{kn} \,]$ and the vector $\mathbf{x}$.

Theorem. Let $\mathbf{A}$ be an $m \times n$ matrix. Then the matrix-vector product $\mathbf{Ax}$ is "linear" in $\mathbf{x}$. That is,

1. $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av}$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and
2. $\mathbf{A}(c\mathbf{u}) = c\mathbf{Au}$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and all c in $\mathbb{R}$.